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A Simple Characterization of Dynamic Completeness in Continuous Time

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Abstract
This paper investigates dynamic completeness of financial markets in which the underlying risk process is a multi-dimensional Brownian motion and the risky securities’ dividends geometric Brownian motions. A sufficient condition, that the instantaneous dispersion matrix of the relative dividends is non-degenerate, was established recently in the literature for single-commodity, pure-exchange economies with many heterogenous agents, under the assumption that the intermediate flows of all dividends, utilities, and endowments are analytic functions. For the current setting, a different mathematical argument in which analyticity is not needed shows that a slightly weaker condition suffices for general pricing kernels. That is, dynamic completeness obtains irrespectively of preferences, endowments, and other structural elements (such as whether or not the budget constraints include only pure exchange, whether or not the time horizon is finite with lump-sum dividends available on the terminal date, etc.).

Keywords: Dynamically-Complete Markets, Geometric Brownian Motion, Asset Pricing.

JEL Classification Numbers: G10, G12.

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1 Introduction

Whether or not a given asset market is dynamically complete is of fundamental importance in financial economics. If the pricing process of the underlying securities is dynamically complete, then options and other derivatives can be uniquely priced by arbitrage arguments and replicated by trading the underlying securities. In the absence of dynamic completeness, however, this is no longer the case; no-arbitrage restrictions do not suffice to guarantee unique option prices while replication may not be possible.

It is crucial, therefore, to be able to associate dynamic completeness with the economic primitives of a given financial environment - in a manner that remains unambiguously verifiable and holds at least generically across the space of these primitives. This is precisely the contribution of the present paper with respect to the case in which the underlying risk process is a Brownian motion and the risky securities’ dividends exponential functions of it.

Specifying the securities’ dividends as geometric Brownian motions has been an important benchmark for the theoretical as well as the applied finance literature. Recently, moreover, it has started featuring prominently also in applied macro- and micro-economic studies. Its popularity rests upon the facilitation of quite realistic financial modeling in which asset prices can be derived in closed form or as solutions to well-known stochastic differential equations. And, as established in the sequel, this lends itself also to theoretical justification. For it allows the property of dynamic completeness to be mapped exclusively to a property of the securities’ dividends.

More precisely, the financial market being dynamically complete can be characterized as the matrix of factor loadings of the relative dividends being nonsingular. Although often asserted implicitly in the relevant literature, this has not been shown explicitly before - at least not to the degree of generality the present study will allow in terms of the supporting economic environment. More importantly perhaps, with respect to the space of the primitive parameters, the characterizing condition is always and easily verifiable while, when it holds, it does so not only generically but universally.

1To name but a few theoretical papers, see Bick [5], Cochrane et al. [11], Constandinides and Zariphopoulou [12], Merton [43]-[44], Oksendal and Sulem [47], Raimondo [53], or Anderson and Raimondo [2]. Applied studies include, for instance, Martens and van Dijk [40], Wong [57], Instefjord [35], Gerber and Shiu [26]-[27], Gatheral and Schied [25], Browne [7], or Biger and Hull [6].

2See, for example, Postali and Picchetti [50], Farhi and Panageas [22], Epaulard and Pommeret [20], Hadjiliadis [28], Hull [34], He [30], Cadenillas and Zapatero [8], Capozza and Kazarian [9], Ericsson [21], Mella-Baral and Perraudin [42], Oren [48], Pennings [49], Promislow and Young [51], Maratha and Ryan [52], Schmidli [55], Milevsky [45], Fleten et al. [24], Deng et al. [15], or Carey and Zilberman [10].

3The typical relative dividend is the dividend of the typical security divided by the dividend of the particular security which has been designated as the numeraire. When the latter security is a money-market account or when its dividend is deterministic, the condition refers to the matrix of factor loadings of the actual risky dividends.

4In most generic results on dynamic completeness, the corresponding condition is shown to hold except for a small set of the primitive parameters. It is nevertheless difficult, if not impossible in some cases, to establish whether it does so for particular values of these parameters. Notable exceptions, of course, are the results in Anderson and Raimondo [1] and in Hugonnier et al. [32].
The relevance of this result becomes evident when viewed in the context of general equilibrium. The typical approach in the literature for obtaining financial equilibria in continuous time has been to compute an Arrow-Debreu equilibrium and use the associated consumption process as pricing kernel in order to construct equilibrium prices for the traded securities. To ensure however that the starting Arrow-Debreu allocation is implementable by trading the given set of securities, their market needs to be dynamically complete. Yet, the equilibrium pricing processes are determined endogenously (via fixed-point arguments) from the model’s primitives (the utility functions of the agents, their endowments, and the dividend processes of the securities) and are expressed as expectations of properly discounted future payoffs. As a result, especially in economies with many heterogeneous agents (increasingly the focus of the asset-pricing literature), and apart from the extremely special cases where one can obtain sufficiently straightforward closed form solutions, verifying from the primitives that the equilibrium pricing process is indeed dynamically complete is a highly non-trivial problem, known as “endogenous completeness.”

Essential progress in this problem was achieved only recently in two important papers, Anderson and Raimondo [1] and Hugonnier et al. [32]. Both papers study a single-commodity, pure-exchange economy with a potentially dynamically complete set of securities and many heterogeneous agents whose preferences over consumption are of the von Neumann-Morgenstern type. In either analysis, the fundamental insight is that the non-degeneracy of the instantaneous dispersion matrix of the relative dividends can be shown to suffice for dynamic completeness and, hence, permit the construction of the equilibrium pricing process via a representative agent. The fundamental assumption is that, with respect to flows during the trading horizon, the securities’ dividends as well as the agents’ utilities and endowments must all be real analytic functions.

Anderson and Raimondo [1] restricts attention to the case in which the underlying risk process is a multi-dimensional Brownian motion while the time-horizon is finite, on the terminal date of which the securities must pay lump-sum dividends. This is because the non-degeneracy condition that is shown to suffice for dynamic completeness is imposed on the lump-sum dividends themselves, at some point in the underlying space. By contrast, Hugonnier et al. [32] allows the underlying risk process to follow a general diffusion and the time-horizon to be infinite. The approach in this paper is such that the non-degeneracy condition can be imposed instead on the flow-dividends, at some point in the underlying space (in a neighborhood of the terminal date when the horizon is finite).

The present paper is a complement to these two studies. It establishes that, when the underlying risk process is a multi-dimensional Brownian motion and the risky dividends are geometric Brownian motions, what amounts to a weaker non-degeneracy condition can be shown to characterize dynamic completeness, using a very different line of proof. In sharp contrast to the methods deployed in

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5See the introductory section in Anderson and Raimondo [1] for an extensive review and discussion of the seminal studies. The existence of the Arrow-Debreu equilibrium itself is due to some assumptions whose form varies in the literature.
Anderson and Raimondo [1] and in Hugonnier et al. [32], the present analysis is based upon completely standard mathematical techniques that do not require analyticity. More importantly, the present results indicate that the relation between dynamic completeness and the non-degeneracy condition in question extends in directions that are rather fundamental for applications.

Indeed, our results are valid for general pricing kernels as long as one of three widely-used in the literature growth conditions is met. They apply, moreover, universally on the underlying state space so that, when dynamic completeness obtains, the instantaneous dispersion matrix of the relative asset prices is non-degenerate everywhere, not almost everywhere. The former property means that, under the risk and dividend specifications in the present paper, our non-degeneracy condition characterizes dynamic completeness irrespectively of the underlying structure for economic activity or the agents’ preferences and endowments. And in conjunction with this, replacing generic non-degeneracy of the asset pricing process with non-degeneracy everywhere can have significant implications for the agents’ portfolio selection problem. It ensures for example that, under mild additional assumptions, in a single-commodity pure-exchange economy the optimal portfolio positions of every agent are everywhere, not almost everywhere, locally bounded.

The remainder of the paper is organized as follows. The next section introduces the theoretical structure under study and analyzes its main elements in the context of the pertinent literature. Section 3 presents the results, which are further interpreted in Section 4. Section 5 concludes, while the Appendix contains proofs and supporting technical material.

2 Setup and Related Literature

In a financial market where trading occurs over a time-interval \( T \subseteq \mathbb{R}_+ \) and the informational structure is given by a standard Brownian motion, well-known no-arbitrage conditions ensure that the securities’ prices are the current expectations of their future dividends valued at some pricing kernel, a strictly-positive one-dimensional Ito process. In what follows, the underlying standard Brownian process will be \( K \)-dimensional (\( K \in \mathbb{N}^* \)), defined on a complete probability space \((\Omega, \mathcal{F}, \pi)\), and depicted as \( \beta: \Omega \times \mathcal{T} \mapsto \mathbb{R}^K \) or \( \beta_k: \Omega \times \mathcal{T} \mapsto \mathbb{R} \) with \( k \in K \equiv \{1, \ldots, K\} \) for the typical dimension. As usual, this is meant to fully describe the exogenous financial risk in the sense that the collection of the sample paths \( \{\beta(\omega, t) : t \in [0, T]\}_{\omega \in \Omega} \) specifies all the distinguishable events.

Given that the underlying uncertainty is driven by Brownian motions, a securities market may be dynamically complete only if the number of securities exceeds that of independent Brownian motions by at least one (i.e only if the market is at least potentially dynamically complete).\(^6\) Hence,

\(^6\)To fix ideas about the underlying concepts, recall that a financial market is said to be complete in a continuous-time setting if it is possible at any \((\omega, t) \in \Omega \times \mathcal{T}\) to instantaneously enter into a portfolio position that will replicate any admissible contingent claim - i.e, any admissible process \( \{Y(\omega, s) : s \in \mathcal{T}\setminus[0, t]\} \). By contrast, the market is said to be dynamically complete if the arbitrary admissible \( \{Y(\omega, s) : s \in \mathcal{T}\setminus[0, t]\} \) can be replicated instead by an admissible self-financing trading strategy (regarding notions of admissibility see, for example, Nielsen [46] §4.1,4.6,5.1,5.4 or Duffie [16] §5.C,6.1). Of course, either definition refers to facilitating the replication of contingent claims. Yet, a
the trading structure will consist of $K + 1$ securities, indexed by $j \in K \cup \{0\}$, which are traded continuously over $\mathcal{T}$. It will be instructive, moreover, to distinguish between two different forms the dividend process of a security may take. Specifically, let $I : \Omega \times \mathcal{T} \mapsto \mathcal{T} \times \mathbb{R}^K$ depict the process \{t, I(\omega, t)\}$_{t \in \mathcal{T}}$. Along the Brownian path \{I(\omega, t)\}$_{t \in \mathcal{T}}$, the typical security may be paying the dividend flow $g_j(I(\omega, \cdot))$ while, if the time-horizon is finite ($\mathcal{T} = [0, T]$ for some $T \in \mathbb{R}^+$), also the lump sum $G_j(I(\omega, T))$ on the terminal date.

Denoting then by $x_s : s \in \mathcal{T} \setminus [0, t]$ the increments’ process $\beta(\cdot, s) - \beta(\cdot, t) : \Omega \mapsto \mathbb{R}^K$ while letting $m : \mathcal{T} \times \mathbb{R}^K \mapsto \mathbb{R}^+$ and $M : \mathcal{T} \times \mathbb{R}^K \mapsto \mathbb{R}^+$ be the pricing kernels, the current price of the typical security $j \in K \cup \{0\}$ can be written as

$$P_j(I(\omega, t)) = \mathbb{E}_\pi \left[ \frac{(MG_j)(T, I(\omega, t), x_T)}{m(I(\omega, t))} + \int_t^T \frac{(mg_j)(s, I(\omega, t), x_s)}{m(I(\omega, t))} ds | \mathcal{F}_t \right]$$

under slight abuse of notation and the proviso that, in the infinite-horizon case ($\mathcal{T} = \mathbb{R}^+$), only the second term on the right-hand side above applies (with $T = \infty$).

In the context of general equilibrium analysis, the pricing kernel cannot be but a weighted average of the individual agents’ equilibrium marginal utilities. In this sense, the essential premise that lies underneath the asset-pricing equation above (and, thus, also behind the analysis that follows) is that utilities, dividends, endowments, and wealth are allowed to be time- as well as state-dependent, as long as this obtains through the realizations of the process $I(\cdot)$. As an approach towards equilibrium asset-pricing theory, this has been the building block for much of the seminal literature.

Obviously, the starting point has been to assume that agents have identical preferences. This has been the launching pad of two related strands of the literature. The first restricts attention to what is essentially the continuous-time analogue of the static (one-period) model: the setting in which the time-horizon is finite and securities pay only lump-sum dividends on the terminal date. The resulting asset-pricing process is given by the first term on the right-hand side of (1) - as in, for example, Bick [4]-[5], He and Leland [29], Raimondo [53], or Anderson and Raimondo [2].

The second approach has been to allow for securities that pay also dividend flows during the time interval while the time-horizon may be infinite. Perhaps the most well-known paper in this strand is Cox et al. [13], the continuous-time analogue of the famous model in Lucas [39], enhanced to include production. In Cox et al. [13], the asset-pricing formula takes the same form as (1), which can be found also in Cochrane et al. [11], Martin [41], Merton [43]-[44], or Wang [56] (whose pricing formula derives actually from Example 3 in Duffie and Skiadas [17]).

Even when the economy consists of agents with heterogenous preferences, the pricing kernel remains a linear function of the equilibrium marginal utilities (the Negishi weights are constant)
if the equilibrium allocation is Pareto-optimal. And again, also in this case, the pricing formula retains the same basic form as in (1) - see, for instance, Anderson and Raimondo [1], Hugonnier et al. [32], Basak and Cuoco [3], Duffie and Zame [18] (see Theorem 1 and the subsequent discussion in Section 5), Dumas [19], Karatzas et al. [38] (see Corollary 10.4), or Riedel [54] (see Theorem 2.1). Clearly, in the context of financial equilibrium, the pricing process under consideration here is quite general, at least as long as Pareto-optimality is a desideratum.\footnote{Equation (1) requires that the pricing kernel is written as an Itô process with respect to $\beta$. It is based, moreover, upon the premise that there are no arbitrage opportunities of any type. With respect to the former requirement, it should be noted that, when the equilibrium allocation is not Pareto-optimal, the representative agent’s utility function will be state-dependent even if all individuals have state-independent preferences and homogenous beliefs (see, for example, Cuoco and He [14]). In fact, the Negishi weights in the construction of the representative agent may even play the role of endogenous state variables which cannot be recovered as functions of the exogenous ones. Regarding the no-arbitrage restriction, there has emerged recently some literature on rational asset-pricing bubbles via the martingale method (see, for example, Hugonnier [33] or Jarrow et al. [36]) as well as on so-called relative arbitrage (see, for instance, Heston et al. [31] or Fernholz and Karatzas [23]), either resting on the very premise that (1) fails.}

Of course, in an equilibrium model, one must choose also a numeraire. Yet, since the underlying informational structure is a filtration, the choice of numeraire here is essentially arbitrary because the equilibrium market-clearing condition will depend only on the relative prices of the traded entities, and will do so node $(\omega, t)$ by node $(\omega, s)$, for $s \neq t$. As a consequence, it is without loss of generality to normalize such that the price of one of the traded entities (typically, one of the commodities) is 1 at all $(\omega, t) \in \Omega \times \mathcal{T}$.

It is also typical in continuous-time models to assume that one of the traded securities (say the zeroth one) is a money-market account, an instantaneously risk-free asset. Alas, this is an endogenous assumption because it restricts directly the market value of this security. Instead, to render the zeroth security instantaneously risk-free, one can simply divide all prices in the model by that of the zeroth security. When one does so, it is now the price of the latter security that is 1 at all $(\omega, t) \in \Omega \times \mathcal{T}$ and, most importantly, this is without any loss of generality (see Anderson and Raimondo [1] for a more detailed discussion). What matter then are the relative prices of the remaining securities, the typical one being $p_n(I(\omega, t)) = P_n(I(\omega, t))/P_0(I(\omega, t)) n \in \mathcal{K}$.

This renormalization turns the buy-and-hold the zeroth security strategy into a trivial money-market account. Which ensures in turn that dynamic completeness is equivalent to the instantaneous dispersion matrix of the relative securities’ prices being almost everywhere non-singular.\footnote{See, for example, Sections 4.1-4.4 and Theorem 5.6 in Nielsen [46].}

That is, the financial market under study here will be dynamically complete if and only if the rank of the Jacobian matrix

$$J_{p}(I(\omega, t)) = \left[ \frac{\partial p_n(I(\omega, t))}{\partial \beta_k(\omega, t)} \right]_{(n,k) \in \mathcal{K} \times \mathcal{K}}$$

is almost everywhere on $\Omega \times \mathcal{T}$ equal to $K$, the number of the independent underlying sources of risk.
Given this, our focus will be on the derivative of the typical relative price with respect to changes in the current realization of the typical Brownian component:

$$\frac{\partial p}{\partial \beta_k} = \frac{\partial p_n(I(\omega, t))}{\partial \beta_k} - p_n(I(\omega, t)) \frac{\partial p_0(I(\omega, t))}{\partial \beta_k} \quad (n, k) \in K \times K$$

with our attention restricted to the dividend processes

$$g_j(I(\omega, t)) = a_j(t) e^{\sigma_j^T \beta(\omega, t)} \quad j \in K \cup \{0\}$$

$$G_j(I(\omega, T)) = A_j(T) e^{\sigma_j^T \beta(\omega, T)} \quad j \in K \cup \{0\}$$

for some deterministic functions $a_j, A_j : \mathcal{T} \mapsto \mathbb{R}_+$ (with $a_j$ strictly positive almost everywhere on $\mathcal{T}$ and $A_j(T) > 0$, unless stated otherwise), and constant factor loadings (instantaneous dispersion) vectors $\sigma_j, \tilde{\sigma}_j \in \mathbb{R}^K$.

Here, the typical terminal dividend is proportional to a $K$-dimensional geometric Brownian motion as long as $\sigma_j \neq 0$ and $A_j(t) = \rho(t) e^{\mu_j t}$ for some $\mu_j \in \mathbb{R} \setminus \{0\}$ and some deterministic supply function $\rho : [0, T] \mapsto \mathbb{R}_+$ (with $\int_0^T \rho(s) \, ds < \infty$). By contrast, $\sigma_j = 0$ renders the dividend riskless and the corresponding security a bond, whose coupon could be, for instance, $A_j(t) = e^{\int_0^t \mu_j(s) \, ds}$ for some deterministic function $\mu_j : [0, T] \mapsto \mathbb{R}_+$ (with $\int_T^0 \mu_j(s) \, ds < \infty$). Similarly, the typical intermediate dividend is proportional to a $K$-dimensional geometric Brownian motion if $\sigma_j \neq 0$ and $a_j(t) = \tilde{\rho}(t) e^{\tilde{\mu}_j t}$ for some $\tilde{\mu}_j \in \mathbb{R} \setminus \{0\}$ and some function $\tilde{\rho} : \mathcal{T} \mapsto \mathbb{R}_+$ (with $\int_0^T \tilde{\rho}(s) \, ds < \infty$). When $\sigma_j = 0$, on the other hand, the corresponding security will be an annuity which may pay, for example, $a_j(t) = e^{\int_0^t \tilde{\mu}_j(s) \, ds}$ for some deterministic function $\tilde{\mu}_j : \mathcal{T} \mapsto \mathbb{R}_+$ (with $\int_T^0 \tilde{\mu}_j(s) \, ds < \infty$).

### 3 Analysis

The component processes $\beta_1, \ldots, \beta_K$ being independent one-dimensional standard Brownian motions, for any $s \in \mathcal{T} \setminus [0, t]$, the increments’ process $\beta(\cdot, s) - \beta(\cdot, t) : \Omega \mapsto \mathbb{R}^K$ is independent of the current filtration $\mathcal{F}_t$, with its realizations distributed $\mathcal{N}(\mathbf{0}, (s-t)\mathbb{I}_K)$. In this case, therefore, the pricing equation (1) reads

$$P_j(I(\omega, t)) = \int_{\mathbb{R}^K} \frac{(MG_j)(T, \beta(\omega, t) + \sqrt{T-t}x)}{m(I(\omega, t))} \, d\Phi(x) \quad j \in K \cup \{0\}$$

$$+ \int_{\mathbb{R}^K} \int_t^T \frac{(mg_j)(s, \beta(\omega, s) + \sqrt{s-t}x)}{m(I(\omega, t))} \, ds \, d\Phi(x)$$

$\Phi$ being the $K$-dimensional standard normal cumulative distribution function.

To study the dispersion of this pricing process with respect to the underlying Brownian process, the subsequent analysis will make use of some functional bounds. These are given by the following growth conditions, all referring to an open ball $B_\delta \subset \mathbb{R}^K$ centered at the origin and of radius $\delta > 0$.
and (apart from case (iii) below) to a parameter $r \in \mathbb{R}$. Specifically, we will say that a function $f : \mathbb{R}^K \mapsto \mathbb{R}$ satisfies

(i) the ordinary growth condition if

$$\exists A \in \mathbb{R}^+ : |f(x)| \leq Ae^{r|x|^2} \text{ a.e. in } \mathbb{R}^K \setminus B_\delta,$$

in which case we will write $f \in \mathcal{G}(r)$,

(ii) the strong growth condition if

$$|f(x)| \leq r + e^{r|x|} \text{ a.e. in } \mathbb{R}^K \setminus B_\delta,$$

written also as $f \in \mathcal{G}^*(r)$,

(iii) the exponential growth condition for $(A, \lambda) \in \mathbb{R}^+ \times \mathbb{R}^K$ if

$$|f(x)| \leq Ae^{\lambda^T x} \text{ a.e. in } \mathbb{R}^K \setminus B_\delta$$

a case that will be denoted by $f \in \mathcal{G}^{**}(A, \lambda)$, and

(iv) the polynomial growth condition if

$$|f(x)| \leq 1 + |x|^r \text{ a.e. in } \mathbb{R}^K \setminus B_\delta$$

denoted also by $f \in \mathcal{P}(r)$.

3.1 Lump-sum Dividends

It will be instructive to begin with the setting in which the time-horizon is finite ($T = [0, T]$ for some $T \in \mathbb{R}^+$) and the securities pay only lump-sum dividends on the terminal date. In this case, the fact that the pricing process is given by the first term on the right-hand side of (5) only (as in, for example, Bick [4]-[5], Anderson and Raimondo [2], Raimondo [53], He and Leland [29]) allows its dispersion with respect to the underlying Brownian process to exhibit integral symmetries that will be fundamental for the present analysis.\footnote{As usual, a.e. means almost everywhere while $| \cdot |$ denotes the Euclidean norm. Notice also that choosing the origin as the center of $B_\delta$ is without any loss of generality. Our definitions could refer instead to $B_\delta$ being centered at an arbitrary $x_0 \in \mathbb{R}^K$. This is because each of the four growth conditions is used only to make inferences about integrals so that such a replacement requires but a trivial change in the variables of integration.}

Moreover, these remain valid irrespectively of the

\footnote{To facilitate parsimonious exposition, often in what follows we will not display the current Brownian node $(\omega, t)$, time, or the value $I(\omega, t)$ as arguments in the corresponding functions. We may write that is $\beta_t$, $P_{jt}$, $p_{nt}$, $m_t$, and $h(s, \beta_t + \sqrt{s-t}x)$ instead of, respectively, $\beta(\omega, t)$, $P_j(I(\omega, t))$, $p_n(I(\omega, t))$, $m(I(\omega, t))$, and $h(s, \beta(\omega, t) + \sqrt{s-t}x)$ for $s \in T \setminus [0, t]$, the latter function standing for $M$, $G_j$, $G_n$, $m$, $g_j$, or $g_n$ depending on the context. Similarly, the expectations may not be shown as conditional even though they are so with respect to the current filtration $\mathcal{F}_t$.}
specification for the pricing kernel as long as an appropriate growth restriction is placed on the latter. More specifically, we will assume that

\[ A_1 \quad M : \{T\} \times \mathbb{R}^K \mapsto \mathbb{R} \text{ is locally integrable and such that } M(T, \cdot) \in \mathcal{G}(r_0) \text{ for some } r_0 \in (0, \frac{1}{2T}). \]

To see the importance of this kernel qualification, fix an arbitrary \( \beta \in \mathbb{R}^K \) and observe that (5) reads

\[
P_j(I(\omega, t)) = \int_{\mathbb{R}^K} \frac{(MG_j)(T, \beta(\omega, t) + \sqrt{T-t}x)}{m(I(\omega, t))} \phi\left(\frac{x - \beta(\omega, t)}{\sqrt{T-t}}\right) dx \quad j \in K \cup \{0\}
\]

where \( \phi \) denotes the standard normal density while the second equality is due to a change in the variables of integration. Notice also that, ignoring the arguments \( T \) and \( \beta(\omega, t) \) to focus solely upon the Brownian increment, the function \( G_j(x) \) satisfies the exponential growth condition for \( |A_n(T)| \) and \( \sigma_n \). Hence, by Lemmas A.1-A.2, it must satisfy also the ordinary growth condition for any \( r_1 > 0 \). Under Assumption A1, therefore, choosing \( r_1 \in (0, \frac{1}{2T} - r_0) \) ensures that the function \( (MG_n)(x) \) satisfies the ordinary growth condition for \( r = r_0 + r_1 \in (0, \frac{1}{2T}) \). And since \( 1/2T \leq 1/\lceil 2(T-t) \rceil \) everywhere on \( T \), we may invoke the following result.

**Claim 3.1** Let \( (\lambda, \beta) \in \mathbb{R}_{++} \times \mathbb{R}^K \) be a parameter vector and suppose that \( f(x) : \mathbb{R}^K \mapsto \mathbb{R} \) is locally integrable and such that \( f \in \mathcal{G}(r) \) for some \( r \in (0, \lambda/2) \). Then, the \( \mathbb{R}^K \mapsto \mathbb{R} \) function

\[
F^{(m)}(\beta) = \int_{\mathbb{R}^K} f(x) \frac{\partial^{\sum_{k=1}^{K} m_k} \phi\left(\sqrt{\lambda}(x - \beta)\right)}{\prod_{k=1}^{K} \partial^{m_k}_{\beta_k}} dx
\]

is well-defined and differentiable with

\[
\frac{\partial F^{(m)}(\beta)}{\partial \beta_k} = \int_{\mathbb{R}^K} f(x) \frac{\partial^{m_k+1+\sum_{l \in K \setminus \{k\}} m_l} \phi\left(\sqrt{\lambda}(x - \beta)\right)}{\partial^{m_k+1}_{\beta_k} \prod_{l \in K \setminus \{k\}} \partial^{m_l}_{\beta_l}} dx \quad k \in K
\]

for any index multiple \( m = (m_1, \ldots, m_K) \in \mathbb{N}^K \).

**Proof.** See Appendix. \( \blacksquare \)

Taking \( \lambda = 1/(T-t) \) and \( m = 0 \in \mathbb{R}^K \) (under the standard convention that the zeroth derivative of a function denotes the function itself), the fundamental pricing equation in (6) refers

\[11\] A function \( f : \mathbb{R}^K \mapsto \mathbb{R} \) is said to be locally integrable if it is Lebesgue measurable and \( \int_V |f(x)| dx < +\infty \) for any compact \( V \subseteq \mathbb{R}^K \).
to a $\mathbb{R}^K \mapsto \mathbb{R}$ function of $\beta$ which is well-defined and differentiable in the typical Brownian dimension with

$$
\frac{m_t \partial P_{jt}}{\partial \beta_{kt}} = (T-t)^{-3/2} \int_{\mathbb{R}^K} (MG_j)(x) (x_k - \beta_{kt}) \phi \left( \frac{x - \beta_t}{\sqrt{T-t}} \right) dx
$$

$$
= (T-t)^{-3/2} \mathbb{E}_x \left[ x_k (MG_j) \left( \beta_t + \sqrt{T-t} x \right) \right]
$$

(7)

for $(j, k) \in K \cup \{0\} \times K$, where $x \sim \mathcal{N}(0, I_K)$, and after changing the variables of integration to obtain the last equality.\textsuperscript{12} It follows that

$$
m_t \sqrt{(T-t)^3} P_{0t}^2 \frac{\partial p_{nt}}{\partial \beta_{kt}} = m_t \sqrt{(T-t)^3} \left( P_{0t} \frac{\partial P_{nt}}{\partial \beta_{kt}} - P_{nt} \frac{\partial P_{0t}}{\partial \beta_{kt}} \right)
$$

$$
= \mathbb{E}_y \left[ M \left( \beta_t + \sqrt{T-t} y \right) G_0 \left( \beta_t + \sqrt{T-t} y \right) \right]
$$

$$
\times \mathbb{E}_x \left[ x_k M \left( \beta_t + \sqrt{T-t} x \right) G_n \left( \beta_t + \sqrt{T-t} x \right) \right]
$$

$$
- \mathbb{E}_x \left[ M \left( \beta_t + \sqrt{T-t} x \right) G_n \left( \beta_t + \sqrt{T-t} x \right) \right]
$$

$$
\times \mathbb{E}_y \left[ y_k M \left( \beta_t + \sqrt{T-t} y \right) G_0 \left( \beta_t + \sqrt{T-t} y \right) \right]
$$

$$
= \mathbb{E}_{(x, y)} \left[ M \left( \beta_t + \sqrt{T-t} x \right) M \left( \beta_t + \sqrt{T-t} y \right) \right]
$$

$$
\times (x_k - y_k)
$$

$$
\times G_0 \left( \beta_t + \sqrt{T-t} y \right) G_n \left( \beta_t + \sqrt{T-t} y \right)
$$

for $(n, k) \in K \times K$ and $x, y \sim \text{i.i.d. } \mathcal{N}(0, (T-t) I_K)$.

Consider now the set

$$
V_0(\beta_t) = \left\{ x \in \mathbb{R}^K : G_0 \left( \beta_t + \sqrt{T-t} x \right) \neq 0 \right\}
$$

Under (4) and at any $(\omega, t) \in \Omega \times T$, this covers $\mathbb{R}^K$ but for a null set. Define also the functions $f_0 \equiv MG_0$ and

$$
\overline{G}_{n} \left( \beta_t + \sqrt{T-t} x \right) = \left\{ \begin{array}{ll}
G_n \left( \beta_t + \sqrt{T-t} x \right) / G_0 \left( \beta_t + \sqrt{T-t} x \right) & \text{if } x \in V_0(\beta_t) \\
0 & \text{otherwise}
\end{array} \right.
$$

We may write then

$$
m_t \sqrt{(T-t)^3} P_{0t}^2 \frac{\partial p_{nt}}{\partial \beta_{kt}} = \mathbb{E}_{(x, y)} \left[ f_0 \left( \beta_t + \sqrt{T-t} x \right) f_0 \left( \beta_t + \sqrt{T-t} y \right) \right]
$$

$$
\times (x_k - y_k) \overline{G}_{n} \left( \beta_t + \sqrt{T-t} y \right)
$$

\textsuperscript{12} Recall footnote 10. All expectation operators in this paper are meant to be conditional on $\mathcal{F}_t$. That is, all variables inside the operator, apart from the normally-distributed vectors with respect to which the expectations are taken, are meant to be fixed at their current values.
which means that any \( \mathbf{v} \in \mathbb{R}^K \) gives
\[
\sum_{k=1}^{K} u_k \frac{\partial p_n ( \mathcal{I} (\omega, t))}{\partial \beta_k (\omega, t)} = \frac{1}{m (\mathcal{I} (\omega, t)) \sqrt{(T - t) \beta_0 (\mathcal{I} (\omega, t))^2}} \times \mathbb{E}_{(x, y)} \left[ f_0(T, \beta (\omega, t) + \sqrt{t-t} \mathbf{x}) f_0(T, \beta (\omega, t) + \sqrt{t-t} \mathbf{y}) \right] 
\]
(8)

Let now \( \sigma_n \equiv \sigma_n - \sigma_0 \) and define the family of hyperplanes
\[
H_{(\rho, \sigma_n)} = \{ \mathbf{x} \in \mathbb{R}^K : \sigma_n' \mathbf{x} = \rho \} \quad \rho \in \mathbb{R}
\]
As this family covers in fact \( \mathbb{R}^K \), for any \( \mathbf{v} \in \mathbb{R}^K \setminus \{0\} \), the hyperplane
\[
H_{(0, \mathbf{v})} = \{ \mathbf{x} \in \mathbb{R}^K : \mathbf{v}' \mathbf{x} = 0 \}
\]
may be written also as
\[
H_{(0, \mathbf{v})} = (\bigcup_{\rho \in \mathbb{R}} H_{(\rho, \sigma_n)}) \cap H_{(0, \mathbf{v})} \\
= \bigcup_{\rho \in \mathbb{R}} \left\{ \mathbf{x} \in \mathbb{R}^K : \left( \begin{array}{c} \sigma_n' \mathbf{x} \\ \mathbf{v}' \mathbf{x} \end{array} \right) = \left( \begin{array}{c} \rho \\ 0 \end{array} \right) \right\} \equiv \bigcup_{\rho \in \mathbb{R}} H_{(\rho, \sigma_n)}
\]

Consider next the line
\[
L(\bar{\mathbf{x}}, \mathbf{v}) = \{ \mathbf{x} \in \mathbb{R}^K : \mathbf{x} = \bar{\mathbf{x}} + r \mathbf{v}, \ r \in \mathbb{R} \}
\]
which passes through \( \bar{\mathbf{x}} \in H_{(0, \mathbf{v})} \) and is parallel to \( \mathbf{v} \). Since \( \mathbf{v} \) and \( H_{(0, \mathbf{v})} \) are not collinear, we have\(^{13}\)
\[
\mathbb{R}^K = \bigcup_{\bar{\mathbf{x}} \in H_{(0, \mathbf{v})}} L(\bar{\mathbf{x}}, \mathbf{v}) = \bigcup_{\rho \in \mathbb{R}} \bigcup_{\bar{\mathbf{x}} \in H_{(\rho, \sigma_n)}} L(\bar{\mathbf{x}}, \mathbf{v})
\]
For any \( \mathbf{x} \in L(\bar{\mathbf{x}}, \mathbf{v}) \), however, \( \bar{\mathbf{x}} \in H_{(\rho, \sigma_n)} \subseteq H_{(0, \mathbf{v})} \) implies that \( \mathbf{v}' \mathbf{x} = r \mathbf{v}' \mathbf{v} \) while \( \mathbf{x}' \mathbf{x} = \bar{\mathbf{x}}' \bar{\mathbf{x}} + r^2 \mathbf{v}' \mathbf{v} \). Letting, therefore, \( H_{(\rho, \sigma_n)} = H_{(\rho, \sigma_n)} \cap V_0 (\beta) \) (and suppressing non-relevant functional arguments), (8) reads
\[
\sum_{k=1}^{K} u_k \frac{\partial p_{nt}}{\partial \beta_{kt}} = \frac{\mathbf{v}' \mathbf{v}}{m t \sqrt{(T - t) \beta_0} \mathbf{p}_0} \int_{\mathbb{R}^2} S (\rho, \rho') \, d\rho d\rho'
\]
(9)
\(^{13}\)Let \( \{ \mathbf{v}_k \}_{k=1}^{K-1} \) be a basis for \( H_{(0, \mathbf{v})} \). As the hyperplane is not collinear with \( \mathbf{v} \), it follows that \( \{ \mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{K-1} \} \) is a basis of \( \mathbb{R}^K \). Therefore, any \( \mathbf{x} \in \mathbb{R}^K \) can be written uniquely as \( \mathbf{x} = \sum_{k=1}^{K-1} r_k \mathbf{v}_k + r \mathbf{v} = \bar{\mathbf{x}} + r \mathbf{v} \) for some \( (r, r_1, \ldots, r_{K-1}) \in \mathbb{R}^K \).
with \( S : \mathbb{R}^2 \mapsto \mathbb{R} \) given by
\[
S(\rho, \rho') = \int_{H_{0,\sigma_n}^r} \int_{\mathbb{R}^2} (r - r') \mathcal{G}_n(\bar{x} + r\mathbf{v}) F_0(\bar{x}, r) F_0(\bar{y}, r') \, dr' \, d\bar{x} \, d\bar{y}
\]
and \( F_0 : \mathbb{R}^{k+1} \mapsto \mathbb{R}_{++} \) defined as
\[
F_0(\bar{x}, r) = (2\pi)^{-K/2} f_0(\bar{x} + r\mathbf{v}) e^{-\frac{\mathbf{v}^T \mathbf{x}_t + 2\mathbf{v}^T \mathbf{y}}{2}}
\]
Of course, the above relation applies in general but for when \( \sigma_n \) and \( \mathbf{v} \) are collinear. In this case, the hyperplanes \( H_{0,\sigma_n}, H_{0,\sigma_n}, \) and \( H_{0,\sigma_n}^v \) all coincide so that
\[
\mathbb{R}^K = \cup_{\bar{x} \in H_{0,\sigma_n}} L(\bar{x}, \mathbf{v})
\]
and, thus,
\[
\sum_{k=1}^K \frac{\partial p_{n_t}}{\partial \beta_{kt}} = \frac{\mathbf{v}^T S(0, 0)}{mt} = \frac{\mathbf{v}^T}{mt \sqrt{(T - t)^3 \mathcal{P}_0^2}}
\]
\[
\int_{H_{0,\sigma_n}^r} \int_{\mathbb{R}^2} (r - r') \mathcal{G}_n(\bar{x} + r\mathbf{v}) F_0(\bar{x}, r) F_0(\bar{y}, r') \, dr' \, d\bar{x} \, d\bar{y}
\]
We are now in position to establish a result which will form the backbone for most of the subsequent analysis.

**Proposition 3.1** Let \( \mathcal{T} = [0, T] \) for some \( T \in \mathbb{R}_{++} \) and suppose that the dividend process of each security \( j \in \mathcal{K} \cup \{0\} \) is given by (3)-(4) with \( a_j(t) = 0 \) a.e. on \([0, T]\). Under \( A1 \) and for all \( \mathbf{v} \in \mathbb{R}^K \setminus \{0\} \), any \( n \in \mathcal{K} \), and all \( (\omega, t) \in \Omega \times \mathcal{T} \), we have

(i) \( (\sigma_n - \sigma_0)^T \mathbf{v} \neq 0 \) only if \( (\sigma_n - \sigma_0)^T \mathbf{v} \sum_{k \in \mathcal{K}} v_k \frac{\partial p_n(\mathcal{T}(\omega, t))}{\partial \beta_{kt}(\omega, t)} > 0 \)

(ii) \( (\sigma_n - \sigma_0)^T \mathbf{v} = 0 \) only if \( \sum_{k \in \mathcal{K}} v_k \frac{\partial p_n(\mathcal{T}(\omega, t))}{\partial \beta_{kt}(\omega, t)} = 0 \)

**Proof.** Under the given dividend specification, the relative terminal dividend is written as
\[
\mathcal{G}_n(\mathcal{T}(\omega, T)) = \mathcal{A}_n(T) e^{\mathcal{V}_n(\omega, T)} \quad n \in \mathcal{K}
\]
where \( \mathcal{A}_n(\cdot) \equiv A_n(\cdot)/A_0(\cdot) \). Suppose first that \( \mathcal{V}_n^T \mathbf{v} \neq 0 \) and let \( t < T \). At any \( (\rho, \rho') \in (\mathbb{R} \setminus \{0\})^2 \), we have
\[
\frac{\mathcal{V}_n^T S(\rho, \rho')}{\mathcal{A}_n(T) e^{\mathcal{V}_n(\omega, T)}} = \int_{H_{0,\sigma_n}^v} \int_{H_{0,\sigma_n}^v} \mathcal{V}_n^T (r - r') e^{\mathcal{V}_n(\bar{x} + r\mathbf{v})} F_0(\bar{x}, r) F_0(\bar{y}, r') \, dr' \, d\bar{x} \, d\bar{y}
\]
\[
eq e^{\rho} \int_{H_{0,\sigma_n}^v} \int_{H_{0,\sigma_n}^v} \mathcal{V}_n^T (r - r') e^{\mathcal{V}_n(\bar{x} + r\mathbf{v})} F_0(\bar{x}, r) F_0(\bar{y}, r') \, dr' \, d\bar{x} \, d\bar{y}
\]
Similarly,

$$\frac{\sigma_h^n v S(0, 0)}{A_n(T) e^{\sigma_\beta n T}} = \int_{H^2_{(\rho, \pi_n)}} \int_{R^2} \sigma_h^n v (r - r') e^{r' \sigma_h^n v} F_0(\bar{x}, r) F_0(\bar{y}, r') drdr'd\bar{x}d\bar{y}$$

And, in either case,

$$\sigma_h^n v \left[ e^{r' \sigma_h^n v} (r - r') + e^{r' \sigma_h^n v} (r' - r) \right] = \left( e^{r' \sigma_h^n v} - e^{r' \sigma_h^n v} \right) \left( \sigma_h^n v r - \sigma_h^n v r' \right) \geq 0$$

the inequality being strict everywhere on $R^2$ but for the null set $\{(r, r') \in R^2 : r = r'\}$. By Lemma A.8, therefore, it must be $\sigma_h^n v S(\rho, \rho') > 0 \forall (\rho, \rho') \in R^2$ so that either of (9)-(10) requires that $\sigma_h^n v \sum_{k=1}^K v_k \frac{\partial p_n}{\partial \beta_k} > 0$.

Let now $\sigma_h^n v = 0$. In this case, only (9) is relevant, for which we have

$$\frac{S(\rho, \rho')}{A_n(T) e^{\sigma_\beta n T}} = e^\rho \int_{H^2_{(\rho, \pi_n)}} \int_{R^2} (r - r') F_0(\bar{x}, r) F_0(\bar{y}, r') drdr'd\bar{x}d\bar{y} = 0$$

the last equality again by Lemma A.8 since

$$e^{r' \sigma_h^n v} (r - r') F_0(\bar{x}, r) F_0(\bar{y}, r') + e^{r' \sigma_h^n v} (r' - r) F_0(\bar{y}, r') F_0(\bar{x}, r) = 0$$

everywhere on $R^{2K} \times R^2$. Clearly, $\sigma_h^n v \sum_{k=1}^K v_k \frac{\partial p_n}{\partial \beta_k} = 0$ now and, as the vector $v$, the node $(\omega, t)$ but also the security $n$ were arbitrarily chosen in either case above, the proposition holds everywhere on $\Omega \times [0, T]$.

To see that this is the case also on $\Omega \times \{T\}$, notice that on the terminal date the price of a security is simply its payoff (appropriately measured in the units of the numeraire). That is, $p_n(I(\omega, T)) = G_n(I(\omega, T))$ for all $n \in \mathcal{K}$ and all $\omega \in \Omega$ and, thus,

$$\sum_{k=1}^K v_k \frac{\partial p_n(I(\omega, T))}{\partial \beta_k(\omega, T)} = \sum_{k=1}^K v_k \frac{\partial G_n(I(\omega, T))}{\partial \beta_k(\omega, T)} = \sigma_h^n v A_n(T) e^{\sigma_\beta n T(\omega, T)}$$

(11)

The required claim is immediate. ■

Consider now the matrix

$$\Sigma_0 = \begin{bmatrix}
\sigma_{11} - \sigma_{01} & \ldots & \sigma_{1K} - \sigma_{0K} \\
\vdots & \ddots & \vdots \\
\sigma_{K1} - \sigma_{01} & \ldots & \sigma_{KK} - \sigma_{0K}
\end{bmatrix}$$

and recall that it is non-singular if and only if there is at least one nonzero entry in the vector $\Sigma_0 v$, for every $v \in R^K \setminus \{0\}$. In light of the preceding result, however, the latter condition is equivalent
to there being at least one nonzero entry in the vector \( J_p(\mathcal{I}(\omega, t)) \mathbf{v} \), for every \( \mathbf{v} \in \mathbb{R}^K \setminus \{0\} \) and at any \((\omega, t) \in \Omega \times [0, T]\). In the current setting, this can be stated formally as follows.

**Corollary 3.1** Let \( \mathcal{T} = [0, T] \) for some \( T \in \mathbb{R}_{++} \) and suppose that the dividend process of each security \( j \in K \cup \{0\} \) is given by (3)-(4) with \( a_j(t) = 0 \) a.e. on \([0, T]\). Under A1, \( \Sigma_0 \) is non-singular if and only if the dispersion matrix of the relative prices \( J_p(\mathcal{I}(\omega, t)) \) is non-singular everywhere on \( \Omega \times \mathcal{T} \).

Notwithstanding its immediacy from Proposition 3.1, yet another (more interesting perhaps) reasoning can be given for the “only if” direction of the non-degeneracy relation between \( \Sigma_0 \) and \( J_p(\mathcal{I}(\omega, t)) \). The corresponding argument underlines the economic intuition and shows that one does not need to assume A1 in this direction. It hinges upon two fundamental properties of the model under study, that the Brownian process fully specifies all financial uncertainty and that, under the given dividend specification, the dimensionality of the dispersion matrix of the relative dividends

\[
J_{\mathcal{T}}(\mathcal{I}(\omega, T)) = \left[ \frac{\partial}{\partial \beta_k(\omega, t)} \left( \frac{G_n(\mathcal{I}(\omega, T))}{G_0(\mathcal{I}(\omega, T))} \right) \right]_{(n,k) \in K \times K}
\]

is inextricably linked to the dimensionality of \( \Sigma_0 \).

The see why the latter is true, let \( L \in K \) and \( \Sigma_0^L \) denote the \( L \)th principal minor of \( \Sigma_0 \). Since the dividend specification under study gives

\[
\frac{\partial}{\partial \beta_k(\omega, t)} \left( \frac{G_n(\mathcal{I}(\omega, T))}{G_0(\mathcal{I}(\omega, T))} \right) = (\sigma_{nk} - \sigma_{0k}) \frac{G_n(\mathcal{I}(\omega, T))}{G_0(\mathcal{I}(\omega, T))} \quad n \in K
\]

the \( L \)th principal minor of \( J_{\mathcal{T}}(\mathcal{I}(\omega, T)) \) is given by

\[
J_{\mathcal{T},L}(\mathcal{I}(\omega, T)) = \left[ \begin{array}{c}
\nabla_{\beta(\omega, t)} \overline{G}_1(\mathcal{I}(\omega, T))^T \\
\vdots \\
\nabla_{\beta(\omega, t)} \overline{G}_L(\mathcal{I}(\omega, T))^T \\
\end{array} \right] = \begin{bmatrix}
(\sigma_{11} - \sigma_{01}) \overline{G}_1(\mathcal{I}(\omega, T)) & \ldots & (\sigma_{1L} - \sigma_{0L}) \overline{G}_1(\mathcal{I}(\omega, T)) \\
\vdots & \ddots & \vdots \\
(\sigma_{L1} - \sigma_{01}) \overline{G}_L(\mathcal{I}(\omega, T)) & \ldots & (\sigma_{LL} - \sigma_{0L}) \overline{G}_L(\mathcal{I}(\omega, T))
\end{bmatrix}
\]

Recall now that, if the matrix \( D \) results from multiplying a column of the square matrix \( C \) by some scalar \( \lambda \), their determinants give \( |D| = \lambda |C| \). Clearly, we must have

\[
\left| J_{\mathcal{T},L}(\mathcal{I}(\omega, T)) \right| = |\Sigma_0^L| \prod_{n=1}^{L} \overline{G}_n(\mathcal{I}(\omega, T)) \quad L \in K
\]
and, thus, $\Sigma_{0}^{L}$ is singular only if so is $J_{\mathcal{F}}(I(\omega, T))$ everywhere on $\Omega \times \{T\}$.

The importance of this observation lies in indicating that, even when $\Sigma_{0}$ is singular, there will still be a sub-collection from the $K$ Brownian dimensions and a sub-collection of securities $K^{*} \subsetneq K$ such that the financial market $K^{*} \cup \{0\}$ ought to be dynamically complete. However, the securities in $K^{*} \cup \{0\}$ generate a filtration that is strictly smaller than $\mathcal{F}$ and the very fact that we choose to model the exogenous financial risk via a Brownian motion of dimensionality $K$ means that, at some $(\omega, t) \in \Omega \times T$, some agent must observe some event that is not measureable with respect to a smaller filtration. That is, regarding the economy as a whole, the informational endowment is indeed the larger filtration $\mathcal{F}$ against which, as shown formally below, the financial market cannot be but dynamically incomplete.

**Corollary 3.2**  Let $\mathcal{T} = [0, T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of each security $j \in K \cup \{0\}$ is given by (3)-(4) with $a_j(t) = 0$ a.e. on $[0, T]$. If $\Sigma_{0}$ is singular, the market is dynamically incomplete.

**Proof.** If $\Sigma_{0}$ is singular, the vectors $\{\sigma_{n} - \sigma_{0}\}_{n=1}^{K}$ are linearly dependent and without loss of generality we may let $X \subseteq \mathbb{R}^{K}$ be their linear span and $L$ its dimension ($L < K$). Furthermore, permuting if necessary the elements of the index set $K$, it is also without loss of generality to take $\{\sigma_{n} - \sigma_{0}\}_{n=1}^{L}$ as an orthonormal basis of $X$. It follows then that the process $\{\beta_{k}\}_{k=1}^{L}$ is a standard $L$-dimensional Brownian motion in its own filtration $\hat{\mathcal{F}} = \left\{\hat{\mathcal{F}}_{t}\right\}_{t \in \mathcal{T}}$ (Levy’s Theorem), with respect to which the financial market is dynamically complete (Corollary 3.1). Yet, $\hat{\mathcal{F}}$ is strictly smaller than $\mathcal{F}$ and, thus, dynamic completeness with respect to the latter filtration is impossible. For, as we have observed above, the dimensionality of $J_{\mathcal{T}}(I(\omega, T))$ is at most $L$ so that the additional $K - L$ securities must be redundant. ■

**Lump-sum dividends, and a money-market account**

To complete our investigation of the setting in which the securities pay only lump-sum dividends on the terminal date, we must also consider the case in which the zeroth security is a money-market account. Of course, the price of such a security is by definition deterministic everywhere on $\Omega \times [0, T]$ and, as a result, the very notion of a dividend for this security becomes meaningless. Nonetheless, it is straightforward to verify that the gist of the preceding analysis remains valid.

This is because, in the presence of a money-market account, we can work with the risk-adjusted probability measure which, under well-known conditions, will still correspond to an informational structure of a $K$-dimensional Brownian motion albeit with a drift.\(^{14}\) That is, save for a deterministic

\(^{14}\)I am referring to the Girsanov theorem (see for instance Karatzas and Shreve [37], §3, Theorem 5.1). The new Brownian motion will be given by $\tilde{\beta}_{t} = \beta_{t} + \int_{0}^{t} Y_{s} ds$ as long as the prices of risk $Y_{s}$ is a $K$-dimensional vector of measurable, adapted processes with respect to the original filtration $\mathcal{F}$, such that $\Pr\left[\int_{0}^{T} Y_{s}^{k} ds < \infty\right] = 1$ for each
It follows that the analysis which established Proposition 3.1 may proceed as before once we replace $G_n$, $A_n$, and $\sigma_n$ by $G_n$, $A_n$, and $\sigma_n$, respectively, and set $f_0(\bar{x}, r) = 1$ on $\mathbb{R}^K \times \mathbb{R}$. The only difference is that $S(\rho, \rho')$ must be replaced in (9)-(10) by

$$S(\rho) = \int_{H_{(\rho, \sigma_n)}^\nu} \int_{\mathbb{R}} rG_n(\bar{x} + rv) F_0(\bar{x}, r) \, dr \, d\bar{x}$$

in this case, with

$$F_0(\bar{x}, r) = (2\pi)^{-K/2} e^{-\frac{\bar{x}^T \bar{x} + 2\nu^T v}{2}}$$

The result follows since $r \sim N(0, 1/(\nu^T \nu))$ and, thus, any $\rho \in \mathbb{R}$ gives

$$\frac{\sigma_n^T \nu S(\rho)}{A_n(T)} e^{\sigma_n^T \beta_t} = e^\rho \int_{H_{(\rho, \sigma_n)}^\nu} \int_{\mathbb{R}} r\sigma_n^T \nu e^{r \sigma_n^T \nu} F_0(\bar{x}, r) \, dr \, d\bar{x} > e^\rho \int_{H_{(\rho, \sigma_n)}^\nu} \int_{\mathbb{R}} r\sigma_n^T \nu F_0(\bar{x}, r) \, dr \, d\bar{x}$$

$$= \frac{e^\rho \sigma_n^T \nu}{\sqrt{2\pi^{K-1}}} \int_{H_{(\rho, \sigma_n)}^\nu} \mathbb{E}[r] e^{-\frac{\bar{x}^T \bar{x}}{2}} d\bar{x} = 0$$

if $\sigma_n^T \nu \neq 0$. Otherwise, for $\sigma_0 = 0$, we have

$$\frac{S(\rho)}{A_n(T)} e^{\sigma_0^T \beta_t} = \frac{1}{\sqrt{2\pi^{K-1}}} \int_{H_{(\rho, \sigma_n)}^\nu} \mathbb{E}[r] e^{-\frac{\bar{x}^T \bar{x}}{2}} d\bar{x} = 0$$

Clearly, Corollary 3.1 may be stated here as follows.

**Corollary 3.3** Let $T = [0, T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of each security $n \in \mathcal{K}$ is given by (3)-(4) with $a_n(t) = 0$ a.e. on $[0, T]$. Suppose also that the zeroth security is a money market account. Under A1 and for all $v \in \mathbb{R}^K \setminus \{0\}$, any $n \in \mathcal{K}$, and all $(\omega, t) \in \Omega \times T$, we have

(i) $(\sigma_n - \sigma_0)^T v \neq 0$ only if $(\sigma_n - \sigma_0)^T v \sum_{k \in \mathcal{K}} v_k \frac{\partial P_n(\xi(\omega, t))}{\partial \beta_k(\omega, t)} > 0$

(ii) $(\sigma_n - \sigma_0)^T v = 0$ only if $\sum_{k \in \mathcal{K}} v_k \frac{\partial P_n(\xi(\omega, t))}{\partial \beta_k(\omega, t)} = 0$

$k \in \mathcal{K}$, and $\exp \left( \sum_{k \in \mathcal{K}} \int_0^t Y^k_s d\beta^k_s + \frac{1}{2} \int_0^t |Y^k_s|^2 ds \right)$ being a martingale. For the latter property, it suffices for example that the Novikov condition $\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^t |Y^k_s|^2 ds \right) \right] < \infty$ holds (see Karatzas and Shreve [37], §3, Corollary 5.13).
Letting, moreover, $\Sigma$ be the matrix $\Sigma_0$ when $\sigma_0 = 0$, Corollaries 3.1-3.2 can be re-written regarding the dispersion matrix of the absolute prices

$$J_P(I(\omega, t)) = \left[ \frac{\partial P_n(I(\omega, t))}{\partial \beta_k(\omega, t)} \right]_{(n,k) \in K \times K}.$$

**Corollary 3.4** Let $T = [0, T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of each security $n \in K$ is given by (3)-(4) with $a_n(t) = 0$ a.e. on $[0, T]$. Suppose also that the zeroth security is a money market account. Under A1, $\Sigma$ is non-singular if and only if $J_P(I(\omega, t))$ is non-singular everywhere on $\Omega \times T$.

**Corollary 3.5** Let $T = [0, T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of each security $j \in K$ is given by (3)-(4) with $a_n(t) = 0$ a.e. on $[0, T]$. Suppose also that the zeroth security is a money market account. If $\Sigma$ is singular, the market is dynamically incomplete.

### 3.2 Dividend Flows

Next, we will examine the setting in which the securities pay only dividend flows during $T$ so that the asset-pricing equation (5) includes only the second term on the right-hand side (as in, for instance, Cochrane et al. [11], Martin [41], or Farhi and Panageas [22]).

As before, our investigation will call for specific analytical manipulations of the fundamental pricing equation. To support their theoretical validity, we will introduce the required conditions on the pricing kernel first for the case in which the time interval is finite: $T = [0, T]$ with $T < \infty$. To this end, recall that the time paths of a Brownian motion being almost surely continuous, for any $\omega \in \Omega$, an arbitrary interval $[t, T] \subseteq T$ will be mapped under $\beta(\omega, \cdot)$ almost surely to a compact subset of $\mathbb{R}^K$. It follows, therefore, that any stochastic process $f(s, \beta_s)$ that is locally integrable with respect to $(s, \beta_s)$ will be bounded on $[t, T]$ and discontinuous at most on a subset of measure zero.\(^{15}\) In other words, $f(s, \beta_s)$ is almost surely integrable on the arbitrary $[t, T]$ and, thus, almost surely locally integrable on $T$.

This means that the time-integrals of the stochastic process can be approximated by Riemann-Stieltjes sums:

$$\int_t^T f(s, \beta_s) \, ds = \lim_{\Pi_t \rightarrow 0} \sum_{i=1}^{\tau} f(s_{i-1}, \beta_{s_{i-1}}) \Delta_i$$

where $\Pi_t = \max_{i=1, \ldots, \tau} \{\Delta_i = s_i - s_{i-1}\}$ denotes the mesh of the typical partition $t = s_0 < s_1 < \cdots < s_{\tau-1} < s_\tau = T$ for some $\tau \in \mathbb{N}^*$ in the approximating sequence. Furthermore, since the

\(^{15}\)Let $a, b \in \mathbb{R}$ with $a < b$ and recall that $f : [a, b] \rightarrow \mathbb{R}^K$ is continuous only if it maps a compact subset of $[a, b]$ to a compact subset of $\mathbb{R}^K$. Moreover, $f$ is integrable if and only if it is bounded and its set of discontinuity points has measure zero (Riemann-Lebesgue theorem).
Brownian increments are independent, for each partition in the approximating sequence we ought to have

\[
E \left[ \sum_{i=1}^{\tau} f(s_{i-1}, \beta_{s_{i-1}}) \Delta_i | F_t \right] = \sum_{i=1}^{\tau} E \left[ f(s_{i-1}, \beta_{s_{i-1}}) | F_t \right] \Delta_i
\]

\[
= \sum_{i=1}^{\tau} E \left[ f \left( s_{i-1}, \beta_t + \sum_{j=0}^{i-1} \beta_{s_j} - \beta_{s_{j+1}} \right) | F_t \right] \Delta_i
\]

\[
= \sum_{i=1}^{\tau} E \left[ f \left( s_{i-1}, \beta_t + \sum_{j=0}^{i-1} \beta_{x_j} \right) | F_t \right] \Delta_i
\]

with the \(x_j\)'s independently distributed \(N(0, \Delta_{s_i})\) and, consequently, \(\sum_{j=0}^{i-1} x_j \sim N(0, (T-t) I_K)\).

As \(\Pi_\tau \to 0\), therefore, we get

\[
E \left[ \int_t^T f(s, \beta_s) \, ds | F_t \right] = \int_t^T E \left[ f(s, \beta_t + x) | F_t \right] \, ds
\]

where \(x \sim N(0, (s-t) I_K)\).

Of course, the stochastic process in question is given here by \(f = mg_j\) and, hence, the need for local integrability refers explicitly to the pricing kernel.\(^{16}\) With this in mind, we will assume that

**A 2** \(m : [0, T] \times \mathbb{R}^K \to \mathbb{R}\) is locally integrable and such that

\[
\forall s \in [0, T], \exists \rho_0 \in \left(0, \frac{1}{2T}\right) : m(s, \cdot) \in G(\rho_0)
\]

In light of the preceding discussion, the first part of assumption A2 means that the pricing equation under study may be written as

\[
P_j(I(\omega, t)) = \int_t^T \frac{E \left[ (mg_j(s, \beta(\omega, t) + x)) | F_t \right]}{m(I(\omega, t))} \, ds = \int_t^T \frac{P_{j,s}(I(\omega, t))}{m(I(\omega, t))} \, ds \quad j \in K \cup \{0\}
\]

where \(P_{j,s}(I(\omega, t))\) is nothing but the absolute price \(P_j(I(\omega, t))\) in the analysis of the preceding subsection if the terminal date \(T\) is replaced by \(s\). The second part of A2 ensures in turn that Claim 3.1 applies to \(P_{j,s}(I(\omega, t))\) (for \(m = 0\)) so that this function as well as its partial derivative with respect to the typical Brownian component are both well-defined. That is,

\[
\frac{\partial P_j(I(\omega, t))}{\partial \beta_k(\omega, t)} = \int_t^T \frac{\partial P_{j,s}(I(\omega, t))}{\partial \beta_k(\omega, t)} \, ds \quad (j, k) \in K \cup \{0\} \times K
\]

which brings us in position to establish our next fundamental result.

\(^{16}\)It is immediate, by its very definition, that \(g_j : [0, T] \times \mathbb{R}^K \to \mathbb{R}\) is locally integrable.
Proposition 3.2 Let $\mathcal{T} \subset \mathbb{R}_+$ and suppose that the dividend process of each security $j \in \mathcal{K} \cup \{0\}$ is given by (3)-(4) with $A_j(T) = 0$. Under $A2$ and for all $\mathbf{v} \in \mathbb{R}^K \setminus \{0\}$, any $n \in \mathcal{K}$, and all $\mathcal{I}(\omega, t) \in \Omega \times \mathcal{T}$, we have

(i) $(\tilde{\sigma}_n - \tilde{\sigma}_0)^\mathbf{T} \mathbf{v} \neq 0$ only if $(\tilde{\sigma}_n - \tilde{\sigma}_0)^\mathbf{T} \mathbf{v} \sum_{k \in \mathcal{K}} v_k \frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} > 0$

(ii) $(\tilde{\sigma}_n - \tilde{\sigma}_0)^\mathbf{T} \mathbf{v} = 0$ only if $\sum_{k \in \mathcal{K}} v_k \frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} = 0$

Proof. Given the preceding arguments, for the typical indices $(j, n, k) \in \mathcal{K} \cup \{0\} \times \mathcal{K} \times \mathcal{K}$, we ought to have

$$P_0 (\mathcal{I}(\omega, t))^2 \frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} = P_0 (\mathcal{I}(\omega, t)) \frac{\partial P_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} - P_n(\mathcal{I}(\omega, t)) \frac{\partial P_0(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)}$$

$$= \int_0^T \left( \mathbb{E}[(m_{g_0}) (s, \beta(\omega, t) + \mathbf{x}) | F_t] \frac{\partial \mathbb{E}(m_{g_0})(s, \beta(\omega, t) + \mathbf{y})| F_t}{\partial \beta_k(\omega, t)} - \mathbb{E}[(m_{g_0}) (s, \beta(\omega, t) + \mathbf{y}) | F_t] \frac{\partial \mathbb{E}(m_{g_0})(s, \beta(\omega, t) + \mathbf{x})| F_t}{\partial \beta_k(\omega, t)} \right) \mathbf{ds}$$

$$= \int_0^T \left( P_{0,s}(\mathcal{I}(\omega, t)) \frac{\partial P_{n,s}(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} - P_{n,s}(\mathcal{I}(\omega, t)) \frac{\partial P_{0,s}(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} \right) \mathbf{ds}$$

$$= \int_0^T P_{0,s}(\mathcal{I}(\omega, t))^2 \frac{\partial P_{n,s}(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} \mathbf{ds} \tag{15}$$

with $P_{j,s}()$ and $p_{n,s}()$ being, respectively, the typical absolute and relative price in the analysis of the preceding subsection taking $s$ to be the terminal date while $\mathbf{x}$ and $\mathbf{y}$ are i.i.d. $\mathcal{N}(0, (s - t) \mathcal{I}_K)$. Recall now the proof of Proposition 3.1. It is trivial to check that replacing $T$, $M$, and $G_j$ with $s$, $m$, and $g_j$, respectively, while letting

$$\tilde{V}_0(s, \beta_t) = \{ \mathbf{x} \in \mathbb{R}^K : g_0(s, \beta_t + \sqrt{s - t} \mathbf{x}) \neq 0 \}$$

(which, for any $s \in \mathcal{T} \setminus [0, t]$, covers $\mathbb{R}^K$ but for a zero-measure set) leads to the conclusion that

(i) $(\tilde{\sigma}_n - \tilde{\sigma}_0)^\mathbf{T} \mathbf{v} \neq 0$ only if $(\tilde{\sigma}_n - \tilde{\sigma}_0)^\mathbf{T} \mathbf{v} P_{0,s}(\mathcal{I}(\omega, t))^2 \sum_{k=1}^K v_k \frac{\partial p_{n,s}(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} > 0$

(ii) $(\tilde{\sigma}_n - \tilde{\sigma}_0)^\mathbf{T} \mathbf{v} = 0$ only if $P_{0,s}(\mathcal{I}(\omega, t))^2 \sum_{k=1}^K v_k \frac{\partial p_{n,s}(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} = 0$

The result follows immediately since the quantity $(\tilde{\sigma}_n - \tilde{\sigma}_0)^\mathbf{T} \mathbf{v}$ is independent of $s$. $lacksquare$

Given this result and the matrix

$$\tilde{\Sigma}_0 = \begin{bmatrix}
\tilde{\sigma}_{11} - \tilde{\sigma}_{01} & \cdots & \tilde{\sigma}_{1K} - \tilde{\sigma}_{0K} \\
\vdots & \ddots & \vdots \\
\tilde{\sigma}_{K1} - \tilde{\sigma}_{01} & \cdots & \tilde{\sigma}_{KK} - \tilde{\sigma}_{0K}
\end{bmatrix}$$

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denoted as \( \tilde{\Sigma} \) when \( \tilde{\sigma}_0 = 0 \), we can immediately derive the appropriate for the current setting re-statements of Corollaries 3.1 and 3.4. In addition, since under the given dividend specification,

\[
\frac{\partial}{\partial \beta_k (\omega, t)} \left( \frac{g_n (I (\omega, s))}{g_0 (I (\omega, s))} \right) = \left( \tilde{\sigma}_{nk} - \tilde{\sigma}_{0k} \right) \frac{g_n (I (\omega, s))}{g_0 (I (\omega, s))} \quad s \in T \setminus [0, t], n \in K
\]

and, hence,

\[
\frac{\partial}{\partial \beta_k (\omega, t)} \int_t^T \frac{g_n (I (\omega, s))}{g_0 (I (\omega, s))} ds = \left( \tilde{\sigma}_{nk} - \tilde{\sigma}_{0k} \right) \int_t^T \frac{g_n (I (\omega, s))}{g_0 (I (\omega, s))} ds \quad n \in K \quad (16)
\]

it is equally straightforward to replicate the argumentation that led to Corollaries 3.2 and 3.3-3.5. Combining then the respective results, we obtain the following statements.

**Corollary 3.6** Let \( T \subseteq \mathbb{R}_+ \) and suppose that the dividend process of each security \( j \in K \cup \{0\} \) is given by (3)-(4) with \( A_j (T) = 0 \). Under \( A_2 \), \( \tilde{\Sigma}_0 \) is non-singular if and only if \( J_p (I (\omega, t)) \) is non-singular everywhere on \( \Omega \times T \). If \( \tilde{\Sigma}_0 \) is singular, moreover, the market is dynamically incomplete, irrespectively of whether or not \( A_2 \) holds.

**Corollary 3.7** When the zeroth security is a money-market account, Proposition 3.2 and Corollary 3.6 apply once \( p_n, \tilde{\Sigma}_0, \) and \( J_p \) are replaced by \( P_n, \tilde{\Sigma}, \) and \( J_P \), respectively.

Turning now to the case in which \( T = \mathbb{R}_+ \), the corresponding results ought to follow from the preceding analysis as \( T \to \infty \). This requires that \( P_{j,s}(\omega,t) \) and \( \frac{\partial P_{j,s}(I(\omega,t))}{\partial \beta_k(\omega,t)} \) are defined at all \( s \in \mathbb{R}_+ \). It presupposes, thus, a strengthening of the pricing kernel condition so that Claim 3.1 applies accordingly (for \( m = 0 \)). To this end, we will assume that

**A 3** \( m : \mathbb{R}_+ \times \mathbb{R}_+^K \to \mathbb{R} \) is locally integrable and such that

\[
m (s, \cdot) \in \mathcal{G} (r) \quad \forall (s, r) \in \mathbb{R}_+ \times \mathbb{R}_+
\]

in order to obtain the following result.

**Proposition 3.3** Proposition 3.2 and Corollaries 3.6-3.7 do hold when \( T = \mathbb{R}_+ \) as long as \( A_2 \) is replaced by \( A_3 \).

**Proof.** Let \( (\omega, t) \in \Omega \times T, n \in K \), as well as \( v \in \mathbb{R}_+^K \{0\} \) be arbitrary. Under condition A3, the analysis that established Proposition 3.2 remains valid for any \( (s, T) \in [t, T] \times (t, \infty) \). As a result, in the current setting, (15) reads

\[
P_0 (I (\omega, t))^2 \frac{\partial p_n (I (\omega, t))}{\partial \beta_k (\omega, t)} = \int_t^\infty P_{0,s} (I (\omega, t))^2 \frac{\partial p_{n,s} (I (\omega, t))}{\partial \beta_k (\omega, t)} ds = \lim_{T \to +\infty} \int_t^T P_{0,s} (I (\omega, t))^2 \frac{\partial p_{n,s} (I (\omega, t))}{\partial \beta_k (\omega, t)} ds
\]
Suppose now that \((\sigma_n - \sigma_0)^T v \neq 0\). We ought to have

\[
P_0(I(\omega, t))^2 (\sigma_n - \sigma_0)^T \sum_{k=1}^K v_k \frac{\partial p_n(I(\omega, t))}{\partial \beta_k(\omega, t)} \geq \int_t^T P_{0,s}(I(\omega, t))^2 (\sigma_n - \sigma_0)^T \sum_{k=1}^K v_k \frac{\partial p_{n,s}(I(\omega, t))}{\partial \beta_k(\omega, t)} ds > 0
\]

the last inequality because

\[
(\sigma_n - \sigma_0)^T \sum_{k=1}^K v_k \frac{\partial p_{n,s}(I(\omega, t))}{\partial \beta_k(\omega, t)} > 0 \quad \forall s \in [t, T], \forall T \in \mathbb{R}^+
\]

by Proposition 3.6(i). Moreover, since

\[
P_0(I(\omega, t))^2 \sum_{k=1}^K v_k \frac{\partial p_n(I(\omega, t))}{\partial \beta_k(\omega, t)} = \int_t^\infty P_{0,s}(I(\omega, t))^2 \sum_{k=1}^K v_k \frac{\partial p_{n,s}(I(\omega, t))}{\partial \beta_k(\omega, t)} ds
\]

part (ii) of Proposition 3.6 also holds. For when \((\sigma_n - \sigma_0)^T v = 0\), the limit above is taken over a sequence of zeros.

Of course, since \(\sum_{k=1}^K v_k \frac{\partial p_n(I(\omega, t))}{\partial \beta_k(\omega, t)}\) is nothing but the nth entry in \(J_p(I(\omega, t))^T v\), it follows immediately that Corollaries 3.6-3.7 remain valid when \(T = \mathbb{R}_+\), as long as A2 is replaced by A3.

### 3.3 Lump-sums and Flows

It remains to examine the case in which the securities may pay both dividend flows during the time-interval as well as lump sums on the terminal date. This presupposes a finite time-horizon and requires that both terms on the right-hand side of (5) apply (as in, for example, Anderson and
Raimondo [1], Hugonnier et al. [32], or Cox et al. [13]). In this case,

\[ P_0 (\mathcal{I}(\omega, t)) \frac{\partial p_n (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} \]

\[ = \frac{\partial P_{n1} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} - \frac{P_n (\mathcal{I}(\omega, t))}{P_0 (\omega, t)} \frac{\partial P_{01} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} \]

\[ + \frac{\partial P_{n2} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} - \frac{P_n (\mathcal{I}(\omega, t))}{P_0 (\omega, t)} \frac{\partial P_{02} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} \]

\[ = \frac{\partial P_{n1} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} - \frac{P_{n1} (\mathcal{I}(\omega, t))}{P_0 (\omega, t)} \frac{\partial P_{01} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} \]

\[ + \frac{\partial P_{n2} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} - \frac{P_{n2} (\mathcal{I}(\omega, t))}{P_0 (\omega, t)} \frac{\partial P_{02} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} \]

\[ - \tilde{P}_n (\mathcal{I}(\omega, t)) \left( \frac{\partial P_{01} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} + \frac{\partial P_{02} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} \right) \]

\[ = P_{01} (\mathcal{I}(\omega, t)) \frac{\partial p_{n1} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} + P_{02} (\mathcal{I}(\omega, t)) \frac{\partial p_{n2} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} \]

\[ - \tilde{P}_n (\mathcal{I}(\omega, t)) \left( \frac{1}{P_{01} (\mathcal{I}(\omega, t))} \frac{\partial P_{01} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} - \frac{1}{P_{02} (\mathcal{I}(\omega, t))} \frac{\partial P_{02} (\mathcal{I}(\omega, t))}{\partial \beta_k (\omega, t)} \right) \]

(17)

where

\[ \tilde{P}_n (\mathcal{I}(\omega, t)) = P_{01} (\mathcal{I}(\omega, t)) P_{02} (\mathcal{I}(\omega, t)) \frac{p_{n2} (\mathcal{I}(\omega, t)) - p_{n1} (\mathcal{I}(\omega, t))}{P_0 (\mathcal{I}(\omega, t))} \]

while \( P_{j1} (\mathcal{I}(\omega, t)) \) and \( P_{j2} (\mathcal{I}(\omega, t)) \) denote, respectively, the first and second term on the right-hand side of (5) with \( p_{ji} (\mathcal{I}(\omega, t)) = \frac{P_{j1} (\mathcal{I}(\omega, t))}{P_{0i} (\mathcal{I}(\omega, t))} \) for \( i = 1, 2 \) being the respective relative prices.

Obviously, the dynamics of the typical relative price in (17) are complex, to the extent that a direct application of our analysis so far concludes only that the market is dynamically complete on \( \Omega \times \{T\} \) if and only if \( \Sigma_0 \) [resp. \( \Sigma \) in the presence of a money-market account] is non-singular. Nonetheless, sufficient conditions for dynamic completeness can be identified also for the interval \([0, T)\) when standard European options are available for trading or when the zeroth security is a money market account.

**Lump-sums and flows, and simple options**

Regarding the former trading setting, suppose that for \( j \in K \cup \{0\} \) a European call on the \( j \)th security with strike price \( P_0 (\mathcal{I}(\omega, t)) \) and maturity date \( T \), as well as an equivalent European put are traded at the node \((\omega, t)\).\(^{17}\) Since the maturity date is the terminal one for the model, the price

\(^{17}\)As usual, European calls and puts are said to be equivalent if they are written on the same underlying, with identical maturity date and exercise price.
of the underlying at maturity will be simply its terminal dividend. Hence, defining the set
\[ V_j (I (\omega, t)) = \{ x \in \mathbb{R}^K : G_j \left( T, \beta (\omega, t) + \sqrt{T-t}x \right) \geq P_0 (I (\omega, t)) \} \]
the current absolute prices for the two options will be given, respectively, by
\[
P_j^C (I (\omega, t)) = \int_{V_j (I (\omega, t))} (MG_j) \left( T, \beta (\omega, t) + \sqrt{T-t}x \right) d\Phi (x) - P_0 (I (\omega, t)) \int_{V_j (I (\omega, t))} M \left( T, \beta (\omega, t) + \sqrt{T-t}x \right) d\Phi (x)
\]
\[
P_j^P (I (\omega, t)) = P_0 (I (\omega, t)) \int_{\mathbb{R}^K \setminus V_j (I (\omega, t))} M \left( T, \beta (\omega, t) + \sqrt{T-t}x \right) d\Phi (x) - \int_{\mathbb{R}^K \setminus V_j (I (\omega, t))} (MG_j) \left( T, \beta (\omega, t) + \sqrt{T-t}x \right) d\Phi (x)
\]
This results in a version of the well-known put-call parity
\[
P_j^P (I (\omega, t)) - P_j^C (I (\omega, t)) = M_0 (I (\omega, t)) P_0 (I (\omega, t)) - P_{j1} (I (\omega, t))
\]
where
\[
M_0 (I (\omega, t)) = \mathbb{E}_x \left[ M \left( t, \beta (\omega, t) + \sqrt{T-t}x \right) \right]
\]
would be the current price of a zero-coupon bond maturing at \( T \), should this be traded. This version of the put-call parity gives rise to the following results.\(^{18}\)

**Proposition 3.4** Suppose that \( \mathcal{T} = [0, T] \) for some \( T \in \mathbb{R}_{++} \) and let the securities’ dividends be given by (3)-(4) for \( j \in \mathcal{K} \cup \{0\} \). Suppose also that European calls with maturity date \( T \) and strike price \( P_0 (I (\omega, t)) \) as well as equivalent puts are traded on every security in \( \mathcal{K} \cup \{0\} \), everywhere on \( \Omega \times [0, T) \). Under A1-A2, \( J_p (I (\omega, t)) \) is non-singular everywhere on \( \Omega \times \mathcal{T} \) if at least one of \( \tilde{\Sigma}_0 \) and \( \Sigma_0 \) is non-singular.

**Proof.** Suppose first that \( \tilde{\Sigma}_0 \) is non-singular. Replace each security \( n \in \mathcal{K} \) with the portfolio that consists of being long on the security and the corresponding European put, and short on the equivalent call and \( M_0 (I (\omega, t)) \) units of the zeroth security. Replace also the zeroth security with the portfolio that consists of being long on \( 1 - M_0 (I (\omega, t)) \) units of the security and the respective European put, and short on the equivalent call. Since the current values of these portfolios are

\(^{18}\)Needless to say, Corollary 3.8 follows immediately from Proposition 3.4. In fact, as dynamic completeness obtains if and only if \( J_p (I (\omega, t)) \) is non-singular almost everywhere on \( \Omega \times \mathcal{T} \), the former claim is equivalent to the latter under the weaker requirement that the requisite European options are available almost everywhere on \( \Omega \times [0, T) \). This notwithstanding, the slightly stronger way in which the proposition is stated in the text has the advantage of offering direct support to Corollary 4.1 and the example to which it refers in the next section.
\( P_{n2}(I(\omega, t)) \) and \( P_{02}(I(\omega, t)) \), respectively, the new Jacobian of relative prices is given by

\[
J_p(I(\omega, t)) = \left[ \frac{\partial p_{n2}(I(\omega, t))}{\partial \beta_k(\omega, t)} \right]_{n \in K}
\]

and dynamic completeness follows immediately from Corollary 3.6.

Suppose next that \( \Sigma_0 \) is non-singular. Replace now each security \( n \in K \) with the portfolio that consists of being long on the corresponding European call and \( M_0(I(\omega, t)) \) units of the zeroth security, and short on the equivalent put. Replace also the zeroth security with the portfolio that consists of being long on \( M_0(I(\omega, t)) \) units of the security and the respective European call, and short on the equivalent put. The current values of these portfolios are \( P_{n1}(I(\omega, t)) \) and \( P_{01}(I(\omega, t)) \), respectively, so that

\[
J_p(I(\omega, t)) = \left[ \frac{\partial p_{n1}(I(\omega, t))}{\partial \beta_k(\omega, t)} \right]_{n \in K}
\]

Dynamic completeness follows now from Corollary 3.1. ■

**Corollary 3.8** Suppose that \( T = [0, T] \) for some \( T \in \mathbb{R}_+ \) and let the securities’ dividends be given by (3)-(4) for \( j \in K \cup \{0\} \). Suppose also that European calls with maturity date \( T \) and strike price \( P_0(I(\omega, t)) \) as well as equivalent puts are traded on every security in \( K \cup \{0\} \), almost everywhere on \( \Omega \times [0, T) \). Under A1-A2, the securities’ market is dynamically complete if at least one of \( \Sigma_0 \) and \( \tilde{\Sigma}_0 \) is non-singular.

Of course, that a financial market may be completed using options is well-known. What may be surprising here is the need for only plain vanilla options. Which implies obviously that the market can be completed also by equally simple forward contracts. For the position of being long on a standard European call and at the same time short on an equivalent put can be replicated by holding an equivalent forward contract. Indeed, a forward contract to buy the typical security at exercise price \( P_0(I(\omega, t)) \) and exercise date \( T \) would be priced at

\[
P_j^F(I(\omega, t)) = \mathbb{E}_x \left[ M(T, \beta(\omega, t) + \sqrt{T-t}x) \right. \\
\left. \left( G_j(T, \beta(\omega, t) + \sqrt{T-t}x) - P_0(I(\omega, t))) \right) \right] \\
= P_{j1}(I(\omega, t)) - M_0(I(\omega, t)) P_0(I(\omega, t)) \\
= P_j^F(I(\omega, t)) - P_j^P(I(\omega, t))
\]

Clearly, the required European options can be replaced in the statements of Proposition 3.4 and Corollary 3.8 by forward contracts with exercise date \( T \) and exercise price \( P_0(I(\omega, t)) \) written on the respective underlying securities in \( K \cup \{0\} \).
Lump-sums and flows, and a money-market account

Proposition 3.4 is a sufficiency result for dynamic completeness in the presence of standard European options or equivalent forward contracts. However, that such securities are traded is no longer needed when the zeroth security is a money market account. In this case, $\frac{\partial P_0(I(\omega,t))}{\partial \beta_k(\omega,t)} = 0 = \frac{\partial P_0(\mathcal{I}(\omega,t))}{\partial \beta_k(\omega,t)}$ everywhere on $\Omega \times [0,T]$ and, thus, (17) reads

$$P_0(t) \frac{\partial p_n(I(\omega,t))}{\partial \beta_k(\omega,t)} = P_{01}(t) \frac{\partial p_{n1}(I(\omega,t))}{\partial \beta_k(\omega,t)} + P_{02}(t) \frac{\partial p_{n2}(I(\omega,t))}{\partial \beta_k(\omega,t)}$$

(18)

where $\frac{\partial p_{ni}(I(\omega,t))}{\partial \beta_k(\omega,t)} = P_{0i}(t)^{-1} \frac{\partial p_{ni}(I(\omega,t))}{\partial \beta_k(\omega,t)}$ for $i = 1, 2$. It follows then from our previous analysis that the $n$th entry of the vector $J_p(I(\omega,t))^\top \mathbf{v}$ will be non-zero if either exactly one of $\sigma_n^1 \mathbf{v}$ and $\sigma_n^2 \mathbf{v}$ is non-zero or both are non-zero yet of the same sign.

This observation allows for Proposition 3.4 to be re-stated, in a way that calls attention to the following subsets of $\mathbb{R}^K \setminus \{0\}$

$$V_0 = \{ \mathbf{v} \in \mathbb{R}^K \setminus \{0\} : \mathbf{v}^\top \sigma_n \tilde{\sigma}_n^\top \mathbf{v} = 0 \ \forall n \in \mathcal{K} \}$$

$$V_+ = \{ \mathbf{v} \in \mathbb{R}^K \setminus \{0\} : \exists n \in \mathcal{K}, \ \mathbf{v}^\top \sigma_n \tilde{\sigma}_n^\top \mathbf{v} > 0 \}$$

**Proposition 3.5** Suppose that $\mathcal{T} = [0, T]$ for some $T \in \mathbb{R}^+$ and let the securities’ dividends be given by (3)-(4) for $j \in \mathcal{K}$, with the zeroth security being a money-market account. Suppose also that $V_0 \cup V_+ = \mathbb{R}^K \setminus \{0\}$. Under $A1-A2$, $J_F(I(\omega,t))$ is non-singular everywhere on $\Omega \times [0,T]$ if at least one of $\Sigma$ and $\tilde{\Sigma}$ is non-singular.

**Proof.** Suppose that at least one of $\Sigma$ and $\tilde{\Sigma}$ is non-singular and consider first an arbitrary $\mathbf{v}_0 \in V_0$. Obviously, it cannot be $\mathbf{v}_0^\top \sigma_n = 0 = \tilde{\sigma}_n^\top \mathbf{v}_0$ for all $n \in \mathcal{K}$ as this would mean that both $\Sigma$ and $\tilde{\Sigma}$ are singular. Hence, $\mathbf{v}_0^\top \sigma_n = 0 \neq \tilde{\sigma}_n^\top \mathbf{v}_0$ or $\mathbf{v}_0^\top \sigma_n \neq 0 = \tilde{\sigma}_n^\top \mathbf{v}_0$ for some $n \in \mathcal{K}$. In the former case, $\sum_{k \in \mathcal{K}} \mathbf{v}_0^k \frac{\partial p_{n1}(I(\omega,t))}{\partial \beta_k(\omega,t)} = 0$ (Proposition 3.1 with $\sigma_0 = 0$) and $\sum_{k \in \mathcal{K}} \mathbf{v}_0^k \frac{\partial p_{n2}(I(\omega,t))}{\partial \beta_k(\omega,t)}$ will have the sign of $\tilde{\sigma}_n^\top \mathbf{v}_0$ (Proposition 3.2 with $\tilde{\sigma}_0 = 0$). If $\mathbf{v}_0^\top \sigma_n \neq 0 \neq \tilde{\sigma}_n^\top \mathbf{v}_0$, on the other hand, $\sum_{k \in \mathcal{K}} \mathbf{v}_0^k \frac{\partial p_{n2}(I(\omega,t))}{\partial \beta_k(\omega,t)}$ will be of the same sign as $\sigma_n^\top \mathbf{v}_0$.

Consider next an arbitrary $\mathbf{v}_0 \in \mathbb{R}^K \setminus (V_0 \cup \{0\})$. By hypothesis, it must be $\mathbf{v}_0 \in V_+$ so that $\mathbf{v}_0^\top \sigma_n \tilde{\sigma}_n^\top \mathbf{v}_0 > 0$ for some $n \in \mathcal{K}$. That is, $\mathbf{v}_0^\top \sigma_n$ and $\tilde{\sigma}_n^\top \mathbf{v}_0$ have the same sign and so must $\sum_{k \in \mathcal{K}} \mathbf{v}_0^k \frac{\partial p_{n1}(I(\omega,t))}{\partial \beta_k(\omega,t)}$ and $\sum_{k \in \mathcal{K}} \mathbf{v}_0^k \frac{\partial p_{n2}(I(\omega,t))}{\partial \beta_k(\omega,t)}$. As this is also the sign of $\sum_{k \in \mathcal{K}} \mathbf{v}_0^k \frac{\partial p_{n2}(I(\omega,t))}{\partial \beta_k(\omega,t)}$, the result follows.

This result replaces the requirement for standard European options with the restriction that the sets $V_0$ and $V_+$ cover $\mathbb{R}^K \setminus \{0\}$. And, as $V_0$ has measure zero, the centre of gravity of this restriction lies in $V_+$ being essentially a covering of $\mathbb{R}^K \setminus \{0\}$, a condition that is guaranteed to hold whenever $\Sigma^\top \tilde{\Sigma}$ is positive semi-definite. To see this, observe that the typical entry in the matrix
\[\Sigma^{-1}\Sigma \text{ is given by } \Sigma^{-1}\Sigma_{ij} = \sum_{n \in K} \sigma_{ni}\sigma_{nj}. \text{ Hence, for any } v \in \mathbb{R}^K \setminus \{0\}, \text{ we have}\]

\[v^\top \Sigma^{-1}\Sigma v = \sum_{i \in K} \sum_{j \in K} v_i \sum_{n \in K} \sigma_{nk}\sigma_{nj} v_j = \sum_{n \in K} \sum_{i \in K} \sigma_{nk} v_i \sum_{j \in K} \sigma_{nj} v_j = \sum_{n \in K} v^\top \sigma_n \tilde{\sigma}_n^\top v\]

which means that, if \(\Sigma^{-1}\Sigma\) is +ve semi-definite, there has to be some \(n \in K\) for which \(\tilde{\sigma}_n^\top v\sigma_n^\top v > 0\).

In other words, \(\Sigma^{-1}\Sigma\) being +ve semi-definite implies that \(V_+ = \mathbb{R}^K \setminus \{0\}\) and permits the preceding result to be stated as follows.

**Corollary 3.9** Suppose that \(T = [0,T]\) for some \(T \in \mathbb{R}_{++}\) and let the securities’ dividends be given by (3)-(4) for \(j \in K\), with the zeroth security being a money-market account. Suppose also that \(\Sigma^{-1}\Sigma\) is +ve semi-definite. Under A1-A2, \(J_P(\mathcal{I}(\omega, t))\) is non-singular everywhere on \(\Omega \times [0,T]\) if at least one of \(\Sigma\) and \(\tilde{\Sigma}\) is non-singular.

Intuitively, the requirement that \(V_+ = \mathbb{R}^K \setminus \{0\}\) guarantees that, for any Brownian realization \(\beta(\omega, t)\), there is some neighborhood of \(\beta(\omega, t)\) in \(\mathbb{R}^K\) and some risky security in the model whose terminal dividend cannot perfectly hedge its dividend flow within the neighborhood, whatever the direction \(v\) in which the Brownian risk process may evolve. Indeed, (12) and (16) imply that the following are equivalent

(i) \(v^\top \sigma_n \tilde{\sigma}_n^\top v > 0\)

(ii) \(v^\top \nabla_{\beta(\omega,t)} G_n(\mathcal{I}(\omega, T)) \nabla_{\beta(\omega,t)} g_n(\mathcal{I}(\omega, s)) v > 0\)

as long as \(g_n(\mathcal{I}(\omega, s)) G_n(\mathcal{I}(\omega, T)) \neq 0\) (which is the case under (3)-(4) for almost all \((\omega, t) \in \Omega \times [0,T]\) and almost all \(s \in (t,T]\).

In fact, a sufficient condition for \(\Sigma^{-1}\Sigma\) to be +ve definite is that

\[\exists \lambda_1, \ldots, \lambda_K \in \mathbb{R}_{++}: \lambda_n \tilde{\sigma}_n = \sigma_n \quad \forall n \in K \quad (19)\]

Which, since \(\lambda_n \nabla_{\beta(\omega,t)} g_n(\mathcal{I}(\omega, s)) = \frac{g_n(\mathcal{I}(\omega, s))}{G_n(\mathcal{I}(\omega, T))} \nabla_{\beta(\omega,t)} G_n(\mathcal{I}(\omega, T))\) whenever \(G_n(\mathcal{I}(\omega, T)) \neq 0\), is equivalent to the requirement that \(\nabla_{\beta(\omega,t)} G_n(\mathcal{I}(\omega, T))\) and \(\nabla_{\beta(\omega,t)} g_n(\mathcal{I}(\omega, s))\) point in the same direction for all \(n \in K\), almost all \((\omega, t) \in \Omega \times [0,T]\), and almost all \(s \in (t,T]\).

In the light of these observations, requiring that \(\mathbb{R}^K \setminus \{0\}\) is covered by \(V_0 \cup V_+\) instead of \(V_+\) alone reflects simply an enhancement of our field of study to account for the cases in which the direction \(v\) may be orthogonal to the vectors \(\nabla_{\beta(\omega,t)} G_n(\mathcal{I}(\omega, T))\) or \(\nabla_{\beta(\omega,t)} g_n(\mathcal{I}(\omega, s))\) for some \(n \in K\), some \((\omega, t) \in \Omega \times [0,T]\), and some \(s \in (t,T]\).

Yet, it also renders necessary that we examine what happens if the complement of \(V_0 \cup V_+\) in \(\mathbb{R}^K \setminus \{0\}\)

\[V_- = \left\{ v \in \mathbb{R}^K \setminus \{0\} : \tilde{\sigma}_n^\top v \sigma_n^\top v \leq 0 \quad \forall n \in K \quad \text{with at least one strict inequality} \right\}\]
is non-empty. For it should be obvious from the preceding discussion, that the latter case allows for directions \( \mathbf{v}_0 \in V_- \) such that \( \mathbf{v}^\top \nabla_{\beta(t)} G_n (\mathcal{I} (\omega, T)) \nabla_{\beta(t)} g_n (\mathcal{I} (\omega, s)) \mathbf{v}_0 < 0 \) for some \( n \in \mathcal{K} \), almost all \((\omega, t) \in \Omega \times [0, T)\), and almost all \( s \in (t, T) \).

Proof. Consider the portfolio that consists of being long on the security and the corresponding European put, and under the weaker requirement that the requisite derivatives are written on every security in \( \emptyset \). Let then \( \sum_{k=1}^{K} v_{0k} \frac{\partial p_{n1}(\omega, t)}{\partial \beta_k(\omega, t)} \) and \( \sum_{k=1}^{K} v_{0k} \frac{\partial p_{n2}(\omega, t)}{\partial \beta_k(\omega, t)} \) will both be non-zero but of opposite sign. It could well be then \( \sum_{k=1}^{K} v_{0k} \frac{\partial p_{n1}(\omega, t)}{\partial \beta_k(\omega, t)} = 0 \) for some \((\omega, t) \in \Omega \times [0, T)\), rendering \( J_p (\mathcal{I} (\omega, t)) \) singular.

This notwithstanding, the financial market can be completed again dynamically if the set of traded securities includes the requisite European options. Yet now these need not be written on every security in \( \mathcal{K} \cup \{0\} \) but only on the securities in the index set

\[
\mathcal{K}^* = \{ n \in \mathcal{K} : \exists \mathbf{v} \in \mathbb{R}^K \setminus \{0\}, \mathbf{v}^\top \sigma_n \delta_n^\top \mathbf{v} < 0 \}
\]

Proposition 3.6 When the zeroth security is a money-market account, Proposition 3.4 applies under the weaker requirement that the requisite derivatives are written on every security in \( \mathcal{K}^* \).

Proof. Obviously, there is nothing to show if \( \mathcal{K}^* = \emptyset \). For this can be only if \( \mathbb{R}^K \setminus (V_0 \cup V_- \cup \{0\}) = \emptyset \). Let then \( \mathcal{K}^* \neq \emptyset \) and suppose first that \( \delta \Sigma \) is non-singular. Replace each security in \( \mathcal{K}^* \) with the portfolio that consists of being long on the security and the corresponding European put, and short on the equivalent call and \( M_0 (\mathcal{I} (\omega, t)) \) units of the zeroth security. Since the current value of this portfolio is given by \( P_{n2} (\mathcal{I} (\omega, t)) \), the new Jacobian of relative prices gives

\[
P_0 (\mathcal{I} (\omega, t)) J_p (\mathcal{I} (\omega, t))^\top \mathbf{v} = \begin{cases} P_{02} (\mathcal{I} (\omega, t)) \sum_{k=1}^{K} v_k \frac{\partial p_{n2}(\omega, t)}{\partial \beta_k(\omega, t)} \quad n \in \mathcal{K}^* \\ P_{01} (\mathcal{I} (\omega, t)) \sum_{k=1}^{K} v_k \frac{\partial p_{n1}(\omega, t)}{\partial \beta_k(\omega, t)} + P_{02} (\mathcal{I} (\omega, t)) \sum_{k=1}^{K} v_k \frac{\partial p_{n2}(\omega, t)}{\partial \beta_k(\omega, t)} \quad n \in \mathcal{K} \setminus \mathcal{K}^* \end{cases}
\]

for the arbitrary \( \mathbf{v} \in \mathbb{R}^K \setminus \{0\} \). Now, without any loss of generality we may take the \( n \)th entry in \( \delta \Sigma \mathbf{v} \) to be non-zero. If \( n \in \mathcal{K}^* \), then the \( n \)th entry of \( J_p (\mathcal{I} (\omega, t))^\top \mathbf{v} \) has the sign of \( \sigma_n \delta_n^\top \mathbf{v} \) (Proposition 3.2). If \( n \notin \mathcal{K}^* \), on the other hand, it must be \( \mathbf{v}^\top \sigma_n \delta_n^\top \mathbf{v} \geq 0 \) leaving, since \( \delta_n^\top \mathbf{v} \neq 0 \) by assumption, two possibilities.

If \( \sigma_n^\top \mathbf{v} = 0 \), the \( n \)th entry of \( J_p (\mathcal{I} (\omega, t))^\top \mathbf{v} \) has again the sign of \( \sum_{k=1}^{K} v_k \frac{\partial p_{n2}(\omega, t)}{\partial \beta_k(\omega, t)} \) (Propositions 3.1-3.2) which is non-zero. Otherwise, \( \mathbf{v}^\top \sigma_n \delta_n^\top \mathbf{v} > 0 \) and its sign is that of the quantity \( \sum_{k=1}^{K} v_k P_{01}(t) \frac{\partial p_{n1}(\omega, t)}{\partial \beta_k(\omega, t)} + P_{02}(t) \frac{\partial p_{n2}(\omega, t)}{\partial \beta_k(\omega, t)} \), a sum of \( K \) pairs of the same (non-zero) sign.

Suppose next that \( \Sigma \) is non-singular. In this case, we replace each security in \( \mathcal{K}^* \) with the portfolio that consists of being long on the corresponding European call and \( M_0 (\mathcal{I} (\omega, t)) \) units of the zeroth security, and short on the equivalent put. The current value of this portfolio is given by
\[ P_{n1}(I(\omega, t)) \] and the new Jacobian of relative prices gives

\[ P_0(I(\omega, t)) J_p(I(\omega, t))^T \mathbf{v} = \begin{cases} P_{01}(I(\omega, t)) \sum_{k=1}^{K} v_k \frac{\partial p_{n1}(I(\omega, t))}{\partial \beta_k(\omega, t)} & n \in \mathcal{K}^* \\ P_{01}(I(\omega, t)) \sum_{k=1}^{K} v_k \frac{\partial p_{n1}(I(\omega, t))}{\partial \beta_k(\omega, t)} + P_{02}(I(\omega, t)) \sum_{k=1}^{K} v_k \frac{\partial p_{n2}(I(\omega, t))}{\partial \beta_k(\omega, t)} & n \in \mathcal{K} \setminus \mathcal{K}^* \end{cases} \]

The balance of the argument proceeds in the same manner as in the previous case.

As a final remark, it should be pointed out that our results for the current setting establish only sufficient conditions for dynamically completeness. Necessity does not follow as immediately under (17) as it did in Sections 3.1-3.2. This is more than evident in the presence of a money-market account, in which case one might be tempted by Corollaries 3.5 and 3.7 to assert that, if \( \tilde{\Sigma} \) and \( \Sigma \) are both singular, the market must be dynamically incomplete under (18). Yet, this may not be true. For it could well be that the kernels

\[ O_\Sigma = \{ \mathbf{v} \in \mathbb{R}^K \setminus \{ \mathbf{0} \} : \Sigma \mathbf{v} = \mathbf{0} \} \quad O_{\tilde{\Sigma}} = \{ \mathbf{v} \in \mathbb{R}^K \setminus \{ \mathbf{0} \} : \tilde{\Sigma} \mathbf{v} = \mathbf{0} \} \]

are both non-empty (i.e., \( \tilde{\Sigma} \) and \( \Sigma \) are both singular) but do not intersect. This, however, being the only case in which the temptation in question leads to fault, we do have the following result.

**Corollary 3.10** Suppose that \( \mathcal{T} = [0, T] \) for some \( T \in \mathbb{R}_{++} \) and let the securities’ dividends be given by (3)-(4) for \( j \in \mathcal{K} \), with the zeroth security being a money-market account. If \( O_\Sigma \cap O_{\tilde{\Sigma}} \neq \emptyset \), the financial market is dynamically incomplete.

In addition, the restriction

\[ \exists \lambda_1, \ldots, \lambda_K \in \mathbb{R} \setminus \{ 0 \} : \lambda_n \tilde{\sigma}_n = \sigma_n \quad \forall n \in \mathcal{K} \quad (20) \]

which is a weakening of (19) and implies that \( \nabla_{\beta(\omega, t)} G_n(I(\omega, T)) \) and \( \nabla_{\beta(\omega, t)} g_n(I(\omega, s)) \) are collinear almost everywhere on \( \Omega \times [0, T] \), means that \( \tilde{\Sigma} \) is non-singular if and only if so is \( \Sigma \) while \( O_\Sigma = O_{\tilde{\Sigma}} \). As a consequence, in the last sentence of each of Propositions 3.5-3.6 as well as of Corollary 3.9, “if at least one of \( \tilde{\Sigma} \) and \( \Sigma \) is non-singular” can be replaced by “if \( \Sigma \) is non-singular.” And in the last corollary above, “if \( O_\Sigma \cap O_{\tilde{\Sigma}} \neq \emptyset \)” can be replaced by “if \( \Sigma \) is singular.”

4 Discussion

We have established that, when the exogenous state variables are Brownian motions and the securities’ dividends are given by (3)-(4), dynamic completeness can be characterized as the matrix of the factor loadings of the relative [resp. actual, if the zeroth security is a bond or a money-market account] dividends being non-degenerate. A condition which, given (12) and (16), must obtain
under the current setting if the dispersion matrix of the relative [resp. actual] dividends is itself non-degenerate.

The latter non-degeneracy condition has been shown to directly suffice for dynamic completeness in two recent seminal papers, by Anderson and Raimondo [1] and by Hugonnier et al. [32]. In either paper, the intuition for the sufficiency argument is essentially the same and hinges fundamentally upon the assumption that all flow primitive variables in the model are real analytic functions. To directly compare with the present analysis, it is easier to focus on the former paper which examines the same setting as in Section 3.3 above but for more general dividend specifications. It argues that the equilibrium pricing process is dynamically complete as follows.\footnote{See in particular Appendices B and D of Anderson and Raimondo [1].}

Let us assume the existence of an open set $V \subseteq \mathbb{R}^K$ s.t. (i) $G_0(I(\omega, T)) > 0$ for all $\beta(\omega, T) \in V$, and (ii) $\exists \omega_0 \in \Omega$ and $\beta(\omega_0, T) \in V$ s.t. $|J_G(\omega_0, T)| \neq 0$. The first condition guarantees of course that the relative terminal dividends are well-defined on $\{T\} \times V$. And so is obviously the dispersion matrix of the relative terminal prices - for $J_p(T, \cdot) = J_G(T, \cdot)$ since there is no value left to a security on the terminal date other than its lump-sum dividend. Yet, $J_p$ is continuous on $\{T\} \times V$ and analytic on $[0, T) \times \mathbb{R}^K$ for the dividend, endowment, and utilities processes are all functions of $I(\omega, s)$ which are analytic on $[0, T) \times \mathbb{R}^K$ and continuous on $\{T\} \times \mathbb{R}^K$. Hence, since analyticity implies continuity, $J_p$ is continuous on $[0, T) \times V$ and condition (ii), which can re-stated as $|J_p(I(\omega_0, T))| \neq 0$, implies in fact that $|J_p(\cdot)| \neq 0$ on $(t_0, T) \times V_0$, for some $t_0$ arbitrarily close to $T$ and $V_0 \subseteq V$ an open neighborhood of $\beta(\omega_0, T)$. However, since a real analytic function is either identically equal to zero or non-zero almost everywhere on its domain, this can be only if $|J_p(\cdot)| \neq 0$ a.e. on $(0, T) \times \mathbb{R}^K$; equivalently, a.e. on $T \times \mathbb{R}^K$ as required.

Of course, the settings studied by Anderson and Raimondo [1], on the one hand, and in Section 3.3 (and, thus, also 3.1) above, on the other, do intersect. As to be expected, therefore, there is a direct relation between the respective sufficient conditions for dynamic completeness. Indeed, under the dividend specifications in (3)-(4), conditions (i)-(ii) above imply also the present non-degeneracy condition. For as the relative dividends are all well-defined everywhere in $V$, by (13), condition (ii) implies that $|\Sigma_0| \neq 0$.

In this sense, our respective non-degeneracy condition follows also from the one assumed by Hugonnier et al. [32], which extends the setting in Anderson and Raimondo [1] to allow for $\beta$ being a general diffusion process as well as for an infinite time-horizon. Their non-degeneracy condition consists also of conditions (i)-(ii), taken exactly as above when the horizon is finite and the securities do pay lump-sum dividends, or with the lump-sum replaced by the flow dividends and $T$ replaced by some intermediate date (sufficiently close to $T$ when the horizon is finite) otherwise. Section 3.2 above being the relevant one in the latter case, given (16), the new version of conditions (i)-(ii) may obtain only if $|\tilde{\Sigma}_0| \neq 0$.\footnote{Hugonnier et al. [32] conduct their study in the presence of a money market account, in which case the requirement for the set $V$ becomes redundant and we may assume directly the existence of a point in the underlying space at
Clearly, when the exogenous state variables are Brownian motions and the securities’ dividends are given by (3)-(4), the sufficient for dynamic completeness non-degeneracy conditions in Anderson and Raimondo [1] as well as in Hugonnier et al. [32] agree with the ones in the present paper, even though the respective supporting arguments differ dramatically. Nonetheless, presenting the non-degeneracy condition as in the present paper has significant advantages within the setting under study. For the non-degeneracy of the matrix of factor loadings entails no real loss of generality as it obtains generically across the space of the model’s primitives.\footnote{Within \( \mathbb{R}^{K^2} \), the space of \( K \times K \) real matrices, the singular ones form a subset of zero-measure.} More importantly perhaps, when it obtains, it applies universally, not generically, on the underlying state space. When dynamic completeness obtains, therefore, the instantaneous dispersion matrix of the relative asset prices is non-degenerate everywhere, not almost everywhere. Which means for instance that, when the pricing kernel results from a single-commodity pure-exchange economy with many heterogenous agents, under mild additional assumptions, the optimal portfolio positions of every agent can be shown to be everywhere, not almost everywhere, locally bounded.\footnote{A function \( f : T \times \mathbb{R}^{K} \) is said to be locally bounded if there exists a neighborhood \( \mathcal{B}_{(t_0, x_0)} \), around any point \( (t_0, x_0) \) in the domain, on which \( f \) is bounded. The property is obviously implied by continuity or (local) integrability.}

To this end, consider a single-commodity pure-exchange economy with \( M \in \mathbb{N}^* \) agents, time-horizon \( T = [0,T] \) for some \( T \in \mathbb{R}_{++} \), and a financial market consisting of \( K + 1 \) securities whose dividends are given by (3)-(4) for \( n \in K \), with the zeroth security being a money market account. Suppose also that the typical agent’s endowment is a deterministic continuous function \( e_m : T \mapsto \mathbb{R}_{+} \) while her utility functions over flow and lump-sum consumption \( u_m, U_m : \mathbb{R}_{+} \mapsto \mathbb{R} \) are both twice continuously differentiable.

Letting the equilibrium asset prices be Itô processes, under the notation we have been using throughout the present study, they evolve as

\[
dP_n (\mathbb{I} (\omega, t)) = \alpha_n (\mathbb{I} (\omega, t)) \, dt + \sum_{k \in K} \frac{\partial P_n (\mathbb{I} (\omega, t))}{\partial \beta_k (\omega, t)} \, d\beta_k (\omega, t) \quad n \in K
\]

for some predictable and Lebesgue-integrable processes \( \alpha_n : \Omega \times T \mapsto \mathbb{R} \). Let also \( \theta_{mj} \) be the typical agent’s portfolio position on the security \( j \in K \cup \{0\} \) and \( \theta_m = (\theta_{m1}, \ldots, \theta_{mK}) \). Due to the which the dividend dispersion matrix is non-degenerate. Yet, their analysis applies also when none of the securities is locally riskless (see their footnote 4), in which case the non-degeneracy condition has to be worded as in the text. The authors show in addition that the dispersion matrix of the dividend flows being non-degenerate may not be necessary for dynamic completeness in some cases. In the light of the present analysis, however, such cases obviously do not include the settings in Sections 3.1-3.2 above or the one referred to by Corollary 3.10.
self-financing condition, her wealth evolves according to the process

\[
dW_m (I(\omega, t), (\theta_m, \theta_m)) = \sum_{n \in K} \theta_{mnt} dP_n (I(\omega, t)) + (\theta_m P_0 + e_m - c) dt
\]

\[
= \left( \sum_{n \in K} \theta_{mnt} \alpha_n (I(\omega, t)) + \theta_m P_0 + e_m - c \right) dt + \sum_{n \in K} \sum_{k \in K} \theta_{mnt} \frac{\partial P_n (I(\omega, t))}{\partial \beta_k} \frac{d\beta_k (\omega, t)}{dt}
\]

Setting \( W_m (I(\omega, t), (\theta_m, \theta_m)) = w \), therefore, the value function of her optimization problem will be given by

\[
V_m (w, t) = \sup_{c, (\theta_m, \theta_m)} \mathbb{E}_x \left[ \int_t^T u_m (c_s) ds + U_m (W_T (w_m, t, c, (\theta_m, \theta_m))) \right]
\]

At any \( \tau \in [t, T] \) and under well-known conditions, this will satisfy the dynamic programming equation

\[
V_m (w, t) = \sup_{c, (\theta_m, \theta_m)} \mathbb{E}_x \left[ \int_t^\tau u_m (c_s) ds + V_m (W_\tau (w_m, t, c, (\theta_m, \theta_m))) \right]
\]

as well as the corresponding Hamilton-Jacobi-Bellman equation

\[
\frac{\partial V_m (w, t)}{\partial t} + \sup_{c, (\theta_m, \theta_m)} \left\{ u_m (c) + AV_m (w, t) \right\} = 0
\]

along with the associated terminal-value condition \( V_m (w, T) = U_m (w) \). Here

\[
AV_m (w, t) = \left( \sum_{n \in K} \theta_{mn} \alpha_n (I(\omega, t)) + P_0 + e - c \right) \frac{\partial V_m (t, w)}{\partial w} + \frac{1}{2} \sum_{k \in K} \sum_{n \in K, n' \in K} \theta_{mn} \theta_{mn'} \frac{\partial P_n (I(\omega, t))}{\partial \beta_k} \frac{\partial P_{n'} (I(\omega, t))}{\partial \beta_k} \frac{\partial^2 V_m (w, t)}{\partial w^2}
\]

and the first-order conditions are given by

\[
u_m' (c^* (w, t)) = \frac{\partial V_m (w, t)}{\partial w}
\]

\[
\alpha_n (I(\omega, t)) \frac{\partial V_m (t, w)}{\partial w} = - \sum_{n' \in K} \sum_{k \in K} \theta_{mn}^* \frac{\partial P_n (I(\omega, t))}{\partial \beta_k} \frac{\partial P_{n'} (I(\omega, t))}{\partial \beta_k} \frac{\partial^2 V_m (w, t)}{\partial w^2} \quad n \in K
\]

\[
\theta_{m0}^* = \frac{W_m (I(\omega, t)) - \sum_{n \in K} \theta_{mn}^* (I(\omega, t)) P_n (I(\omega, t))}{P_0 (I(\omega, t))}
\]

Letting then \( \alpha (I(\omega, t))^T = (\alpha_1 (I(\omega, t)), \ldots, \alpha_K (I(\omega, t))) \), the equations indexed by \( n \in K \) above
can be written compactly as
\[
\alpha (I(\omega, t)) \frac{\partial V_m(t, w)}{\partial w} = -J_p(I(\omega, t)) J_p(I(\omega, t))^T \theta_m^* \frac{\partial^2 V_m(t, w)}{\partial w^2}
\]

Hence, as long as the dispersion matrix \(J_p(I(\omega, t))\) is invertible, the typical agent’s optimal portfolio positions are given by
\[
\theta_m^*(I(\omega, t)) = -\frac{\partial V_m(t, w)}{\partial w} [J_p(I(\omega, t)) J_p(I(\omega, t))^T]^{-1} \alpha (I(\omega, t))
\]
\[
\theta_{0m}^*(I(\omega, t)) = W_m(I(\omega, t)) - \sum_{n \in K} \theta_{mn}^*(I(\omega, t)) P_n(I(\omega, t)) \frac{P_0(t)}{P_0(t)}
\]

Observe now that, as long as \(J_p(I(\omega, t))\) is continuous, all entries in the expression for \(\theta_m^*(I(\omega, t))\) above, apart possibly from \(\alpha (I(\omega, t))\), will be continuous. The latter quantity, moreover, is a vector of integrable and, thus, locally bounded processes. Clearly, \(\theta_m^*(I(\omega, t))\) is locally bounded at \((\omega, t)\). And so is \(\theta_{m0}^*(I(\omega, t))\) given that the asset prices but also the agent’s wealth are all Itô and, hence, continuous processes.

Clearly, to complete the argument it remains to establish sufficient conditions for \(J_p(I(\omega, t))\) to be non-singular and continuous everywhere on \(\Omega \times T\). Which is what the following claim does.

**Corollary 4.1** Let \(T = [0, T]\) for some \(T \in \mathbb{R}_{++}\) and the securities’ dividends be given by (3)-(4) for \(n \in K\) with the zeroth security being a money market account. Let this be the financial market of a single-commodity, pure-exchange economy with \(M \in \mathbb{N}^*\) agents, whose endowments are continuous deterministic functions and whose utilities, as functions of the underlying Brownian risk process, satisfy

(i) \(\forall m \in \{1, \ldots, M\}, \exists \bar{r}_m, r_m > 0\) s.t. \(u'_m \in \mathcal{P}(\bar{r}_m)\) and \(U'_m \in \mathcal{P}(r_m)\)

Let also the dividends’ factor loadings be such that

(ii) at least one of \(\bar{\Sigma}\) and \(\Sigma\) is non-singular, and

(ii.a) \(V_0 \cup V_+ = \mathbb{R}^K \setminus \{0\}\), or

(ii.b) European calls with maturity date \(T\) and strike price \(P_0(t)\) as well as equivalent puts are freely marketed on every security in \(K^*\), at every \((\omega, t) \in \Omega \times [0, T]\).

Then, the equilibrium portfolio positions of every agent are locally bounded everywhere on \(\Omega \times T\).

---

24 It is standard practice in the literature to assume that the value function is everywhere three times differentiable with respect to wealth so that \(\frac{\partial^3 V_m(w, t)}{\partial w^3}\) is everywhere continuous.

25 In the light of our analysis in Sections 3.1-3.2, Corollary 4.1 applies also in those settings under in fact simplified versions of conditions (i)-(ii) above. Namely, when the risky securities pay only lump-sum dividends, (i) refers only to \(U'_m\) while (ii) to the non-degeneracy of \(\Sigma\) with (ii.a)-(ii.b) removed. When \(T = \mathbb{R}_+\) and, thus, the securities pay only dividend-flows, (i) refers to \(u'_m\) and (ii) to the non-degeneracy of \(\bar{\Sigma}\) with (ii.a)-(ii.b) removed.
Proof. Observe first that, in equilibrium, the aggregate consumption in this economy must equal the aggregate endowment. Hence,

\[
\sum_{m=1}^{M} c^*_m(x) = \begin{cases} 
\sum_{m=1}^{M} \left( \tilde{e}_{ms} + \sum_{n \in K} \theta^*_m \theta_{mns} g_n \left( \beta_t + \sqrt{s-t} \bar{x} \right) \right) & s \in (t, T) \\
\sum_{m=1}^{M} \left( e_{mT} + \sum_{n \in K} \theta^*_m \theta_{mns} g_n \left( \beta_t + \sqrt{T-t} \bar{x} \right) \right) & \end{cases}
\]

Yet, the given dividend specifications mean that \( g_n(\beta_t, \cdot, \cdot) \in \mathcal{G}^* (|a_n|, \tilde{\sigma}_n) \) and \( G_n(\beta_t, \cdot) \in \mathcal{G}^* (|A_n|, \sigma_n) \) for all \( n \in K \). Which implies in turn that \( g_n(\beta_t, \cdot) \in \mathcal{G}^* (\tilde{r}_n) \) and \( G_n(\beta_t, \cdot) \in \mathcal{G}^* (r_n) \) for some \( \tilde{r}_n, r_n > 0 \) (Lemma A.1). And as also trivially \( e_{ms} \in \mathcal{G}^* (e_{ms}) \), Lemma A.5 requires that each bracketed term in either summation on the right-hand side above satisfies the strong growth condition for some \( \tilde{r}_m > 0 \) and \( r'_m > 0 \), respectively. By the same lemma, moreover, so does either sum itself. In other words, \( \sum_{m=1}^{M} c^*_m \) and, thus, also \( c^*_m \) satisfy the strong growth condition for some \( r_0 > 0 \) and for any \( m \in M \). Given condition (i) then, the individual marginal utilities satisfy \( u'_m, U'_m \in \mathcal{G} (r) \) for any \( r > 0 \) (Lemma A.6).

As long as the market is dynamically complete, however, these marginal utilities give rise to the equilibrium pricing kernels via a representative agent. Specifically, we ought to have

\[
m(w, t) = \sum_{m=1}^{M} \tilde{\lambda}_m \frac{\partial V_m(w, t)}{\partial w} = \sum_{m=1}^{M} \tilde{\lambda}_m u'_m(c^*_m(w, t))
\]

\[
M(w, T) = \sum_{m=1}^{M} \tilde{\lambda}_m \frac{\partial V_m(w, T)}{\partial w} = \sum_{m=1}^{M} \lambda_m U'_m(c^*_m(w, T))
\]

where \( \lambda_m, \tilde{\lambda}_m \in [0, 1] \) for all \( m \) with \( \sum_{m=1}^{M} \tilde{\lambda}_m = 1 = \sum_{m=1}^{M} \lambda_m \) are fixed weights.

It follows then that either pricing kernel must satisfy assumption A3 (Lemma A.7), and thus also A1-A2. Given condition (ii), therefore, Propositions 3.5-3.6 guarantee that the financial market is indeed dynamically complete. More importantly, that the dispersion matrix \( J_p(I(\omega, t)) \) is invertible everywhere on \( \Omega \times T \). All of its entries being, moreover, differentiable with respect to the typical Brownian realization (Claim 3.1 for \( \mathbf{m} = (1, \ldots, 1) \in \mathbb{R}^K \)), it is also continuous everywhere on \( \Omega \times T \). □

5 Concluding Remarks

In an Arrow-Debreu economy, the agents may shift consumption or income across states and time by trading a complete set of contingent claims, once and for all at the beginning of time. When they are instead constrained to trade a given set of securities, the market is said to be dynamically complete if repeated trading of the securities can still deliver any allocation that would be feasible under a complete set of contingent claims. Under continuous-time trading, this may be possible by trading a finite set of securities rapidly enough, even though the information about the state of
the world is revealed through a stochastic process. In particular, when the underlying uncertainty is driven by Brownian motions, this can happen if the securities market is potentially dynamically complete (i.e., the number of securities exceeds that of independent Brownian motions by at least one). \(^{26}\)

Yet, potential dynamic completeness does not suffice by itself. Some form of independence amongst the securities’ payoffs must obtain in addition. In general, once the securities’ prices are appropriately deflated, this refers to the non-degeneracy of their instantaneous dispersion with respect to the underlying stochastic process. For which, in turn, the present paper shows that the linear independence of the factor loadings of the relative dividends suffices when the dividends are specified as in (3)-(4).

As a result, this is related to the literature via two recent seminal papers, Anderson and Raimondo [1] and Hugonnier et al. [32]. The latter being a significant extension of the former, both papers show that the non-degeneracy of the dividends’ dispersion matrix suffices for dynamic completeness in a single-commodity, pure-exchange economy with many heterogeneous agents if the intermediate flows of all dividends, utilities, and endowments are analytic functions. In striking contrast, especially on account of the simplicity of the mathematical technique, the present argument does not use the analyticity assumption anywhere. In fact, it explicitly calls for its violation when the trading setting is the one Anderson and Raimondo [1] focuses upon. Indeed, dynamic completeness in this case is achieved via the use of simple European options. And derivative securities in general have non-analytic payoff functions.

Furthermore, the present analysis refers to a general pricing kernel, applying thus irrespectively of preferences, endowments, and other structural elements (such as whether the agents’ budget constraints include only pure exchange, whether or not the time horizon is finite, whether or not lump-sum dividends are available on the terminal date etc.), as long as some rather standard growth conditions are met. \(^{27}\) Equally importantly perhaps, the present sufficiency condition ensures the universal non-degeneracy of the asset-prices’ dispersion matrix. And as we saw in the preceding section, by means of an example embedded in the settings of the aforementioned papers, the distance between generic non-degeneracy and non-degeneracy at every point can be of fundamental importance for applications.

Of course, attention here was restricted to a very specific functional form for dividends and to the exogenous risk process being Brownian. And even though this combination has been used extensively in the literature, it does mean that the present results do not extend (at least not readily) to a larger class of models (such as, for example, Ornstein-Uhlenbeck or more generally affine processes) that is becoming increasingly the forefront of the financial economics literature.

\(^{26}\)When the underlying stochastic process is not Brownian, the required number of securities may be larger.

\(^{27}\)For example, our strong growth condition, which implies the validity of assumption A3 (Lemma A.2) and, thus, also of A1-A2, is a local (around the origin) version of the growth condition in Anderson and Raimondo [1]. Our ordinary growth condition, on the other hand, is a standard tool for the study of the heat equation (see, for example, Nielsen [46] Appendix C).
Nevertheless, the setting of the present study has always been an important theoretical benchmark in the quest for fundamental equilibrium insight. In this sense, the importance of being able to determine explicitly if and when a pricing process is dynamically complete in this setting is obvious. Especially when given in a manner that remains unambiguously verifiable and holds generically across the space of the primitives.

References


A Appendix

Lemma A.1 Let $h : \mathbb{R}^K \rightarrow \mathbb{R}$ be such that $h \in G^* (A, \lambda)$ for some $(A, \lambda) \in \mathbb{R}_{++} \times \mathbb{R}^K$. Then $h \in G^* (r_0)$ for some $r_0 > 0$.

Proof. Recall first that, for any $x \in \mathbb{R}^K$, we have

$$|\lambda^\top x| = |\lambda| \times |x| \times |\cos \theta (\lambda, x)| \leq |\lambda| \times |x|$$

where $\theta (\lambda, x)$ denotes the angle between the vectors $\vec{\lambda}$ and $\vec{x}$. Observe next that, by hypothesis, there must be $\delta > 0$ such that $|h (x)| \leq Ae^{\lambda^\top x}$ for almost all $x \in \mathbb{R}^K \setminus B_\delta$. We ought to have, therefore,

$$Ae^{\lambda^\top x} \leq Ae^{\ln A + |\lambda| \times |x|} \leq e^{2|\lambda| \times |x|} \leq e^{r_0 |x|} < r_0 + e^{r_0 |x|}$$

for any $x \in \mathbb{R}^K : |x| \geq \ln A / |\lambda|$ and any $r_0 \geq 2|\lambda|$. Letting then $\delta_0 = \max \{\delta, \ln A / |\lambda|\}$ gives

$$|h (x)| < r_0 + e^{r_0 |x|} \quad \text{a.e. in } x \in \mathbb{R}^K \setminus B_{\delta_0}$$

and the result follows.  

Lemma A.2 Let $h : \mathbb{R}^K \rightarrow \mathbb{R}$ be such that $h \in G^* (r_0)$ for some $r_0 > 0$. Then $h \in G (r)$ for all $r > 0$.

Proof. Consider an arbitrary $r > 0$. By hypothesis, there ought to be some $r_0, \delta_0 > 0$ such that

$$|h (x)| \leq r_0 + e^{r_0 |x|}$$

for almost all $x \in \mathbb{R}^K \setminus B_{\delta_0}$. Yet, for all $x \in \mathbb{R}^K : |x| \geq r_0 / r$, we have

$$r_0 + e^{r_0 |x|} \leq r_0 + e^{r |x|^2} \leq (1 + r_0) e^{r |x|^2}$$

Hence, letting $\delta = \max \{\delta_0, r_0 / r\}$ gives

$$|h (x)| \leq (1 + r_0) e^{r |x|^2} \quad \text{a.e. in } \mathbb{R}^K \setminus B_\delta$$

and the result follows.  

38
Lemma A.3 Let \((\lambda, \beta) \in \mathbb{R}^+ \times \mathbb{R}^K\) be a parameter vector and consider the functions \(\phi\left(\sqrt{\lambda}(x - \beta)\right): \mathbb{R}^K \mapsto \mathbb{R}^+\) and \(f(x): \mathbb{R}^K \mapsto \mathbb{R}\). If \(f \in \mathcal{G}(r)\) for some \(r \in (0, \lambda/2)\), then \(f\phi \in \mathcal{G}(r_1)\) for some \(r_1 < 0\).

Proof. By hypothesis, there exist \((A, r, \delta) \in \mathbb{R}^+ \times (0, \lambda/2) \times \mathbb{R}^+\) such that

\[
|f(x)| \leq Ae^{r|x|^2} \quad \text{a.e. in } \mathbb{R}^K \setminus B_\delta
\]

Take then some \(\alpha > 0\) and let \(\delta_1 = \max\{\delta, |\beta|/\alpha\}\). For any \(x \in \mathbb{R}^K: |x| \geq \delta_1\), we have \(|x^\top \beta| \leq |x| \times |\beta| \leq \alpha |x|^2\) and, thus,

\[
r|x|^2 - \frac{\lambda}{2}|x - \beta|^2 = \left(r - \frac{\lambda}{2}\right) |x|^2 - \frac{\lambda}{2} |\beta|^2 + \lambda x^\top \beta
\]

\[
\leq \left(r - \frac{\lambda}{2}\right) |x|^2 + \lambda |x|^2 \leq \left(r + \lambda \left(\alpha - \frac{1}{2}\right)\right) |x|^2
\]

Choosing, therefore, \(\alpha \in (0, \frac{\lambda - 2r}{2\lambda})\) establishes that

\[
\left|f(x) \phi\left(\sqrt{\lambda}(x - \beta)\right)\right| = \frac{1}{\sqrt{2\pi}} |f(x)| e^{-\frac{1}{2}|x|^2} \leq \frac{A}{\sqrt{2\pi}} e^{r_1 |x|^2}
\]

for almost all \(x \in \mathbb{R}^K \setminus B_{\delta_1}\) and where \(r_1 = r + \lambda \left(\alpha - \frac{1}{2}\right) < 0\). The result follows. □

Lemma A.4 Suppose that \(h: \mathbb{R}^K \mapsto \mathbb{R}\) is such that

\[
|h(x)| \leq Ae^{r|x|^2} \quad \text{a.e. in } \mathbb{R}^K \setminus B_\delta
\]

for some \((A, \delta) \in \mathbb{R}_+^2\) and some \(r < 0\). Then,

\[
\exists C > 0: \quad |h(x)| \prod_{k=1}^K \max\{1, x_k^2\} \leq C \quad \text{a.e. in } \mathbb{R}^K \setminus B_\delta
\]

Proof. By hypothesis, there exist \(A, \delta > 0\) and \(r < 0\) such that

\[
|h(x)| \prod_{k=1}^K \max\{1, x_k^2\} \leq Ae^{r|x|^2} \prod_{k=1}^K \max\{1, x_k^2\}
\]

for almost all \(x \in \mathbb{R}^K: |x| \geq \delta\). Yet, the right-hand side quantity above is continuous and vanishes as \(\min_{k \in \mathcal{K}} |x_k| \rightarrow \infty\). These properties require, respectively, the existence of \((c, \varepsilon) \in (0, \infty) \times [\delta, \infty)\)
and of \( x_0 \in \overline{B}_\varepsilon \) such that

\[
Ae^{r_1|x|^2} \prod_{k=1}^{K} \max \{1, x_k^2\} \leq \begin{cases} 
Ae^{r_1|x_0|^2} \prod_{k=1}^{K} \max \{1, x_{0k}^2\} & \forall x \in \mathbb{R}^K \setminus \overline{B}_\varepsilon \\
Ae^{r_1|x_0|^2} \prod_{k=1}^{K} \max \{1, x_{0k}^2\} & \forall x \in \overline{B}_\varepsilon 
\end{cases}
\]

where \( \overline{B}_\varepsilon \) denotes the closure in \( \mathbb{R}^K \) of \( B_\varepsilon \). It follows, therefore, that

\[
|h(x)| \prod_{k=1}^{K} \max \{1, x_k^2\} \leq \begin{cases} 
Ae^{r_1|x_0|^2} \prod_{k=1}^{K} \max \{1, x_{0k}^2\} & \text{a.e. in } \mathbb{R}^K \setminus \overline{B}_\varepsilon \\
Ae^{r_1|x_0|^2} \prod_{k=1}^{K} \max \{1, x_{0k}^2\} & \text{a.e. in } \overline{B}_\varepsilon \cap \mathbb{R}^K \setminus B_\delta 
\end{cases}
\]

That is,

\[
|h(x)| \prod_{k=1}^{K} \max \{1, x_k^2\} \leq C \quad \text{a.e. in } \mathbb{R}^K \setminus B_\delta
\]

where \( C = A \max \left\{ c, e^{r_1|x_0|^2} \prod_{k=1}^{K} \max \{1, x_{0k}^2\} \right\} \).

**Proof of Claim 3.1**

Fix an arbitrary \( \beta \in \mathbb{R}^K \) and an arbitrary dimension \( k \in \mathcal{K} \). Our argument, which will focus upon the interval \((\beta_k - \epsilon_k, \beta_k + \epsilon_k)\) for some \( \epsilon_k > 0 \), will be presented in steps, the first being merely a remark regarding a well-known property of the standard normal density.

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**Step 1.** Recall that the zeroth derivative \((m = 0)\) refers to the function \( \phi \) itself. In this case, defining the constant function \( H_0 : \mathbb{R} \mapsto \{1\} \) and taking \( C_0 = 1 \), the only thing that changes in the proof below is the order of the first three steps. Specifically, the proof begins from step 3 with steps 1-2 following as they are now needed only for the differentiability part.

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**Proof of Claim 3.1**

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28 Recall that the zeroth derivative \((m = 0)\) refers to the function \( \phi \) itself. In this case, defining the constant function \( H_0 : \mathbb{R} \mapsto \{1\} \) and taking \( C_0 = 1 \), the only thing that changes in the proof below is the order of the first three steps. Specifically, the proof begins from step 3 with steps 1-2 following as they are now needed only for the differentiability part.

29 To lessen the notational burden of our exposition, in what follows, we will abuse notation slightly using \( \phi \) to denote the standard normal density function regardless of the dimensionality of its domain. Of course, the latter will always be obvious.
and, hence,

\[ |H_n(x)| \leq \sum_{i=0}^{n} |c_i|x^i \leq C_n|x|^n \]

for any \( x \in \mathbb{R} \setminus (-1, 1) \) and for \( C_n = \sum_{i=0}^{n} |c_i| \). Observe also that

\[ |x|^n < e^n|x| \leq e^{\frac{2n^2}{2}} \]

the second inequality being valid for any given \( \alpha > 0 \) and all \( x \in \mathbb{R} : |x| \geq 2n/\alpha \). Furthermore, for any \( \tilde{z} \in (-\epsilon, \epsilon) \), we have

\[ \sqrt{\lambda}|x_k - \beta_k - \tilde{z}_k| \geq \sqrt{\lambda}(|x_k| - |\beta_k| - |\tilde{z}_k|) > \sqrt{\lambda}(|x_k| - |\beta_k| - \epsilon_k) \geq \sqrt{\lambda}(|x_k| - |\beta_k| - 1) \geq \frac{2n}{\alpha} \geq 1 \]

as long as \( \epsilon_k \leq 1, |x_k| \geq 1 + \frac{2m}{\alpha\sqrt{\lambda}} + |\beta_k| \), and \( \alpha \leq 2n \).

Step 2. It follows from the preceding step that

\[ \frac{\partial^{\sum_{k=1}^{K} m_k}(\sqrt{\lambda}(x - \beta))}{\prod_{k=1}^{K} \partial(\beta)^{m_k}} = \prod_{k=1}^{K} H_{m_k}(\sqrt{\lambda}(x_k - \beta_k)) \phi(\sqrt{\lambda}(x_k - \beta_k)) \]

and

\[ |H_{m_k}(\sqrt{\lambda}(x_k - \beta_k - \tilde{z}_k))| \phi(\sqrt{\lambda}(x_k - \beta_k - \tilde{z}_k)) \leq C_{m_k} e^{-\frac{\lambda(1-\alpha)(x_k - \beta_k - \tilde{z}_k)^2}{2}} = C_{m_k} \phi(\sqrt{\lambda}(1-\alpha)(x_k - \beta_k - \tilde{z}_k)) \]

for some \( C_{m_k} > 0 \), all \( x_k \in \mathbb{R} : |x_k| \geq 1 + \frac{2m_k}{\alpha\sqrt{\lambda}} + |\beta_k| \), any \( \alpha \in (0, 2m_k] \), any \( \tilde{z}_k \in (-\epsilon, \epsilon) \), and any \( \epsilon \in (0, 1) \). Letting, therefore, \( \tilde{z}_k = 0 \) for all \( k \in K \), \( \overline{m} = \max \{m_k : k \in K\} \), and \( \beta_0 = \max \{\|\beta_k\| : k \in K\} \) means that

\[ \left| \frac{\partial^{\sum_{k=1}^{K} m_k}(\sqrt{\lambda}(x - \beta))}{\prod_{k=1}^{K} \partial(\beta)^{m_k}} \right| \leq \prod_{k=1}^{K} C_{m_k} \phi(\sqrt{\lambda}(1-\alpha)(x_k - \beta_k)) \]

for any \( x \in \mathbb{R}^K \setminus B_{\delta_0} \) where \( \delta_0 = 1 + \frac{2m_0}{\alpha\sqrt{\lambda}} + \beta_0 \).

Step 3. To show now that \( F^{(m)}(\cdot) \) is well-defined set \( m_0 = \min \{m_k : k \in K\} \) and choose \( \alpha \in (0, \min \{\frac{L-2r}{\lambda}, 2m_0\}) \) above to ensure that \( r < \frac{\lambda(1-\alpha)}{2} \). This allows us to use Lemma A.3, which
ensures the existence of $A, r_1, \delta_1 > 0$ such that

$$\left| f(x) \frac{\partial^{\Sigma_{k=1}^K m_k} \phi \left( \sqrt{\lambda} (x - \beta) \right)}{\prod_{k=1}^K \partial^{\delta_{mk}}_k} \right| \leq \frac{A}{2\sqrt{\pi}} \prod_{k=1}^K C_{mk} e^{-r_1|x|^2} \text{ a.e. in } \mathbb{R}^K \setminus B_\delta$$

taking $\delta = \max \{\delta_0, \delta_1\}$. It follows then that

$$\int_{\mathbb{R}^K \setminus B_\delta} \left| f(x) \frac{\partial^{\Sigma_{k=1}^K m_k} \phi \left( \sqrt{\lambda} (x - \beta) \right)}{\prod_{k=1}^K \partial^{\delta_{mk}}_k} \right| \, dx \leq \frac{A}{\sqrt{2\pi}} \prod_{k=1}^K C_{mk} \int_{\mathbb{R}^K \setminus B_\delta} e^{-r_1|x|^2} \, dx < \frac{A}{\sqrt{2r_1}} \prod_{k=1}^K C_{mk}$$

Yet, the functions $f, \phi$, and $H_{mk}$ for any $k \in K$ are all locally integrable with respect to $x$. That is,

$$\int_{B_\delta} |f(x)| \frac{\partial^{\Sigma_{k=1}^K m_k} \phi \left( \sqrt{\lambda} (x - \beta) \right)}{\prod_{k=1}^K \partial^{\delta_{mk}}_k} \, dx = c_1$$

for some $c_1 \in \mathbb{R}^+$ and, hence,

$$\left| \int_{\mathbb{R}^K} f(x) \frac{\partial^{\Sigma_{k=1}^K m_k} \phi \left( \sqrt{\lambda} (x - \beta) \right)}{\prod_{k=1}^K \partial^{\delta_{mk}}_k} \, dx \right| \leq \int_{B_\delta} |f(x)| \frac{\partial^{\Sigma_{k=1}^K m_k} \phi \left( \sqrt{\lambda} (x - \beta) \right)}{\prod_{k=1}^K \partial^{\delta_{mk}}_k} \, dx + \int_{\mathbb{R}^K \setminus B_\delta} f(x) \frac{\partial^{\Sigma_{k=1}^K m_k} \phi \left( \sqrt{\lambda} (x - \beta) \right)}{\prod_{k=1}^K \partial^{\delta_{mk}}_k} \, dx$$

$$< c_1 + \frac{A}{\sqrt{2r_1}} \prod_{k=1}^K C_{mk} < \infty$$

Step 4. Regarding next differentiability observe that, given the first equality in step 2 and letting
\( g_m(k) := H_{mk}(\cdot) \phi(\cdot) \) for \( k \in \mathcal{K} \) (to economize on the length of exposition), we have

\[
\left| \frac{F'(m)(\beta_k + z_k, \beta_{-k}) - F'(m)(\beta)}{z} \right| = \int_{\mathbb{R}^K} \left| f(x) \left[ \prod_{l \in \mathcal{K}\setminus\{k\}} g_{ml} \left( \frac{\sqrt{\lambda}(x_l - \beta_l)}{z_k} \right) \times \left( g_{mk}(\sqrt{\lambda}(x_k - \beta_k)) - g_{mk}(\sqrt{\lambda}(x_k - \beta_{-k})) - \frac{\partial g_{mk}(\sqrt{\lambda}(x_k - \beta_k))}{\partial \beta_k} \right) \right] \right| \, dx \\
\leq \int_{\mathbb{R}^K} \left| f(x) \prod_{l \in \mathcal{K}\setminus\{k\}} g_{ml} \left( \frac{\sqrt{\lambda}(x_l - \beta_l)}{z_k} \right) \times \left( g_{mk}(\sqrt{\lambda}(x_k - \beta_k)) - g_{mk}(\sqrt{\lambda}(x_k - \beta_{-k})) - \frac{\partial g_{mk}(\sqrt{\lambda}(x_k - \beta_k))}{\partial \beta_k} \right) \right| \, dx \\
= \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}^K} \left| f \left( \frac{x - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K}\setminus\{k\}} g_{ml} (x_l - \beta_l) \left| \frac{\partial g_{mk}(x_k - \beta_k - \gamma z_k)}{\partial \beta_k} - \frac{\partial g_{mk}(x_k - \beta_k)}{\partial \beta_k} \right| \, dx \\
\leq \frac{|\gamma z_k|}{\sqrt{\lambda}} \int_{\mathbb{R}^K} \left| f \left( \frac{x - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K}\setminus\{k\}} g_{ml} (x_l - \beta_l) \frac{\partial^2 g_{mk}(x_k - \beta_k - \gamma z_k)}{\partial \beta_k^2} \right| \, dx \\
= \left| \frac{\gamma z_k}{\sqrt{\lambda}} \right| \int_{\mathbb{R}^K} \left| f \left( \frac{x - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K}\setminus\{k\}} g_{ml} (x_l - \beta_l) \frac{\partial^2 g_{mk}(x_k - \beta_k - \gamma z_k)}{\partial \beta_k^2} \right| \, dx \\
= \frac{|z|}{\sqrt{\lambda}} \int_{\mathbb{R}^K} \left| f \left( \frac{x - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K}\setminus\{k\}} g_{ml} (x_l - \beta_l) \frac{\partial^2 g_{mk}(x_k - \beta_k - \gamma z_k)}{\partial \beta_k^2} \right| \, dx \\
\leq \frac{|z|}{\sqrt{\lambda}} \int_{\mathbb{R}^K} \left| f \left( \frac{x - \beta}{\sqrt{\lambda}} + \beta \right) \prod_{l \in \mathcal{K}\setminus\{k\}} g_{ml} (x_l - \beta_l) \frac{\partial^2 g_{mk}(x_k - \beta_k - \gamma z_k)}{\partial \beta_k^2} \right| \, dx \tag{22}
\]

for any \( z_k \in (-\epsilon_k, \epsilon_k) \setminus \{0\} \) and where the third and fourth equality above follow from the mean-value theorem and apply for \( \gamma, \rho \in (0, 1) \) while the second and the last equality are due to a change in the variables of integration (done purely to simplify the presentation). Needless to say, to establish differentiability, it suffices to show that the quantity on the right-hand side of (22) vanishes as \( |z| \to 0 \).

Step 5. Recall now that, by hypothesis, \( |f(x)| \leq Ae^{r|x|^2} \) for some \( (A, r) \in \mathbb{R}^{++} \times (0, \lambda/2) \) and a.e.
in \( \mathbb{R}^K \setminus \mathcal{B}_{\delta_1} \) for some \( \delta_1 > 0 \). Notice also that, as \( |z_k| \to 0 \), we have

\[
\lim_{|z_k| \to 0} \left| \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda}(x_l - \beta_l) \right) g_{m_{k+2}} \left( \sqrt{\lambda}(x_k - \beta_k - \gamma \delta z) \right) \right| = \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda}(x_l - \beta_l) \right) g_{m_{k+2}} \left( \sqrt{\lambda}(x_k - \beta_k) \right) \leq C_{m_{k+2}} \prod_{l \in \mathcal{K} \setminus \{k\}} C_{m_l} \phi \left( \sqrt{\lambda(1-\alpha)}(x - \beta) \right)
\]

the inequality being valid for all \( x \in \mathbb{R}^K : |x| \geq \delta_0 \) where \( \delta_0 \) and \( \alpha \) have been defined already in Step 2.

Let now \( \delta_2 = \max \{ \delta_0, \delta_1 \} \). In conjunction with Lemma A.3, Lemma A.4 ensures the existence of \( C > 0 \) such that

\[
\int_{\mathbb{R}^K \setminus \mathcal{B}_{\delta_2}} |f(x)| \phi \left( \sqrt{\lambda(1-\alpha)}(x - \beta) \right) dx \leq AC \int_{\mathbb{R}^K \setminus \mathcal{B}_{\delta_2}} \prod_{k=1}^{K} \frac{dx}{\max \{ 1, x_k^2 \}} \leq AC \int_{\mathbb{R}^K} \prod_{k=1}^{K} \frac{dx}{\max \{ 1, x_k^2 \}} = 4^K AC
\]

the equality due to the fact that the \( x_k \)'s are independently distributed and, thus,

\[
\int_{\mathbb{R}^K} \prod_{k=1}^{K} \frac{dx}{\max \{ 1, x_k^2 \}} = \prod_{k=1}^{K} \int_{\mathbb{R}} \frac{dx_k}{\max \{ 1, x_k^2 \}} = \prod_{k=1}^{K} \left( \int_{-1}^{1} dx_k + 2 \int_{1}^{+\infty} x_k^{-2} dx_k \right)
\]

**Step 6.** Observe now that (22) can be re-written as

\[
\left| \frac{F^{(m)}(\beta_k + z, \beta_k)}{z} - \int_{\mathbb{R}^K} f(x) \left( \partial_{\beta_k}^{m_{k+1}} \prod_{l \in \mathcal{K} \setminus \{k\}} m_l \phi \left( \sqrt{\lambda}(x - \beta) \right) \right) dx \right| < |z| \int_{\mathbb{R}^K \setminus \mathcal{B}_{\delta_2}} \left| \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda}(x_l - \beta_l) \right) g_{m_{k+2}} \left( \sqrt{\lambda}(x_k - \beta_k - \gamma \delta z_k) \right) dx \right|
\]

\[
+ |z| \int_{\mathcal{B}_{\delta_2}} \left| \prod_{l \in \mathcal{K} \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda}(x_l - \beta_l) \right) g_{m_{k+2}} \left( \sqrt{\lambda}(x_k - \beta_k - \gamma \delta z_k) \right) dx \right|
\]

We have just shown that, as \( |z| \to 0 \), the first term on the right-hand side of this inequality is dominated by the quantity \( 4^K AC C_{m_{k+2}} \prod_{l \in \mathcal{K} \setminus \{k\}} C_{m_l} |z| \), which vanishes. For the second term, we
have

\[
\lim_{|z_k| \to 0} \int_{B_{|z_k|}} f(x) \prod_{l \in K \setminus \{k\}} g_{m_l} \left( \sqrt{\lambda(x_l - \beta_l)} \right) g_{m_{k+2}} \left( \sqrt{\lambda(x_k - \beta_k - \gamma \delta z_k)} \right) \, dx
\]

\[
= \int_{B_{|z_k|}} f(x) \prod_{l \in K \setminus \{k\}} H_{m_l} \left( \sqrt{\lambda(x_l - \beta_l)} \right) H_{m_{k+2}} \left( \sqrt{\lambda(x_k - \beta_k)} \right) \phi \left( \sqrt{\lambda}(x - \beta) \right) \, dx
\]

\[
= c_2 \in \mathbb{R}_{++}
\]

the last equality since the integrand is locally integrable. As \(|z_k| \to 0\), therefore, the term in question becomes \(c_2 |z_k|\) which vanishes as well. To complete the proof, let \(\epsilon_k \to 0\). \(\blacksquare\)

**Lemma A.5** Let \(N \in \mathbb{N}^*\) and for all \(n = 1, \ldots, N\) suppose that \(h_n: \mathbb{R}^K \to \mathbb{R}\) is such that \(h_n \in \mathcal{G}^*(r_n)\) for some \(r_n > 0\). Then, for any \((\theta_1, \ldots, \theta_N) \in \mathbb{R}^N \setminus \{0\}\), it must be \(\sum_{n=1}^N \theta_nh_n \in \mathcal{G}^*(r_0)\) for some \(r_0 > 0\).

**Proof.** By hypothesis, there are \((\delta_1, \ldots, \delta_N) \in \mathbb{R}_{++}^N\) such that

\[
|h_n(x)| \leq r_n + e^{r_n|x|} \quad \text{a.e. in } \mathbb{R}^K \setminus B_{\delta_n}, \quad n = 1, \ldots, N
\]

Let then \(\delta = \max \{1, \delta_1, \ldots, \delta_N\}\). For almost all \(x \in \mathbb{R}^K \setminus B_{\delta},\) we ought to have

\[
\left| \sum_{n=1}^N \theta_nh_n(x) \right| \leq \sum_{n=1}^N |\theta_n| |h_n(x)| \leq \sum_{n=1}^N |\theta_n| \left( r_n + e^{r_n|x|} \right)
\]

\[
\leq N \left( \theta r + \theta e^{r|x|} \right)
\]

\[
= N\theta r + e^{\ln N + \ln \theta + r|x|}
\]

\[
\leq N\theta r + e^{(\ln N + \ln \theta + r)|x|} \leq r_0 + e^{r_0|x|}
\]

where \(\theta = \max_{n=1,\ldots,N} |\theta_n|, \quad r = \max_{n=1,\ldots,N} r_n, \quad \text{and} \quad r_0 = \max \{N\theta r, \ln N + \ln \theta + r\}\). The result follows. \(\blacksquare\)

**Lemma A.6** Let \(f: \mathbb{R} \to \mathbb{R}\) and \(h: \mathbb{R}^K \to \mathbb{R}\) be such that \(f \in \mathcal{P}(r_1)\) and \(h \in \mathcal{G}^*(r_0)\) for some \(r_0, r_1 \in \mathbb{R}_{++}\). Then \(f \circ h \in \mathcal{G}(r)\) for all \(r > 0\).

**Proof.** By hypothesis, there must be some \(\delta_0 > 0\) such that

\[
|h(x)| \leq r_0 + e^{r_0|x|}
\]

for almost all \(x \in \mathbb{R}^K \setminus B_{\delta_0}\). Which, by Lemma A.2, means in turn that

\[
|h(x)| \leq (1 + r_0) e^{r_0|x|^2}
\]
for any \( \tilde{r} > 0 \) and almost all \( x \in \mathbb{R}^K \setminus B_{\delta} \) where \( \tilde{\delta} = \max \{ \delta_0, r_0/\tilde{r} \} \). Fix now an arbitrary \( r > 0 \).

Again by hypothesis, there must exist some \( \delta_1 > 0 \) such that
\[
|f(y)| \leq 1 + |y|^{r_1}
\]
for almost all \( y \in \mathbb{R} \setminus (-\delta_1, \delta_1) \). Letting, therefore, \( \tilde{r} = r/r_1 \) and \( \delta = \max \{ \tilde{\delta}, \delta_1 \} \) it must be
\[
|f(h(x))| \leq 1 + |h(x)|^{r_1} = 1 + (1 + r_0)^{r_1} e^{r |x|^2} \leq e^{r |x|^2}
\]
almost everywhere in \( \mathbb{R}^K \setminus B_\delta \). The result follows.

**Lemma A.7** Let \( M \in \mathbb{N}^\ast \) and for all \( m = 1, \ldots, M \) suppose that \( U_m : \mathbb{R}^K \rightarrow \mathbb{R} \) is such that \( U_m \in G(r_m) \) for some \( r_m > 0 \). Then, for any \( (\lambda_1, \ldots, \lambda_M) \in [0,1]^M \) with \( \sum_{m=1}^M \lambda_m = 1 \), it must be\( \sum_{m=1}^M \lambda_m U_m \in G(\max_{m=1,\ldots,M} r_m) \). Furthermore, if \( U_m \in G(r) \) for all \( r > 0 \) and all \( m \), then \( \sum_{m=1}^M \lambda_m U_m \in G(r) \) for all \( r > 0 \).

**Proof.** By hypothesis, there must be \( (\delta_1, \ldots, \delta_M) \in \mathbb{R}_{+}^M \) such that
\[
|f_m(x)| \leq A_m e^{r_m |x|^2} \quad \text{a.e. in } \mathbb{R}^K \setminus B_{\delta_m} \quad m = 1, \ldots, M
\]
Defining then \( \delta = \max \{ \delta_1, \ldots, \delta_M \} \), for almost all \( x \in \mathbb{R}^K \setminus B_\delta \), we ought to have
\[
\left| \sum_{m=1}^M \lambda_m f_m(x) \right| \leq \sum_{m=1}^M \lambda_m A_m e^{r_m |x|^2} \leq \left( \sum_{m=1}^M \lambda_m A_m \right) e^{r |x|^2}
\]
where \( r = \max_{m=1,\ldots,M} r_m \) as required. The remainder claim follows immediately if one sets \( r_m = r \) for all \( m \).

**Lemma A.8** Let \( S \subseteq \mathbb{R}^K \) be of non-zero Lebesgue measure. Suppose also that the functions \( f : S \times S \rightarrow \mathbb{R}_+ \) and \( h : S \times S \rightarrow \mathbb{R} \) are such that
\((i)\) \( f(x,y) = f(y,x) \) a.e. on \( S \times S \),
\((ii)\) \( h(x,y) + h(y,x) \geq 0 \) a.e. on \( S \times S \), and
\((iii)\) \( fh(\cdot) \) is Lebesgue-integrable over \( S \times S \).

Then
\[
\int_{S \times S} f(x,y) h(x,y) \, d(x,y) \geq 0
\]
with strict inequality iff \( f(x,y)[h(x,y) + h(y,x)] \neq 0 \) on a subset of \( S \times S \) of positive Lebesgue measure.
Proof. Since $hf$ is integrable, by the Fubini-Tonelli theorem, the integral in question can be written as an iterated one:

$$\int_{S \times S} f(x,y) h(x,y) \, d(x,y) = \int_S \left( \int_S f(x,y) h(x,y) \, dy \right) \, dx$$

and, by re-naming the variables of integration, we can write it also as

$$\int_{S \times S} f(x,y) h(x,y) \, d(x,y) = \int_{S \times S} f(y,x) h(y,x) \, d(y,x)$$

$$= \int_S \left( \int_S f(y,x) h(y,x) \, dy \right) \, dx$$

Hence,

$$2 \int_{S \times S} f(x,y) h(x,y) \, d(x,y)$$

$$= \int_S \left( \int_S f(x,y) h(x,y) \, dy \right) \, dx + \int_S \left( \int_S f(y,x) h(y,x) \, dy \right) \, dx$$

$$= \int_S \left( \int_S f(x,y) [h(x,y) + h(y,x)] \, dy \right) \, dx \geq 0$$

Obviously, the inequality is strict iff $f(x,y) [h(x,y) + h(y,x)] \neq 0$ on a subset of $S \times S$ of positive measure. ■